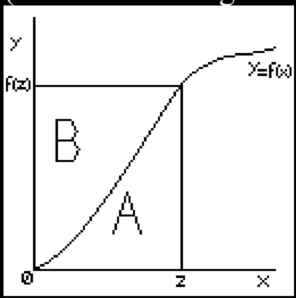
Note on a Method for Integrating Certain Functions

Outline

The fundamental theorem of calculus states that the area under a curve y=f(x) from x=a to x=b can be found by integrating f(x) with respect to x, over the limits a to b. However, we can also apply this to the area *to the left of* the curve y=f(x) from x=a to x=b; we simply integrate $x=f^{-1}(y)dy$ from f(a) to f(b).

This, then, means that the areas A and B in the diagram below can be found by integration of, respectively, f and f^{-1}.

(We are assuming in the diagram that f(0)=0 for simplicity, but this is not in fact necessary)



More specifically, $A=\int_0^z f(x)dx$, while $B=\int_{f(0)}^{f(z)} f^{-1}(y)dy$.

But A+B has area (z*f(z)) - (0*f(0)) = z*f(z).

Therefore the sum of the expressions for A and B must be z*(f(z)-f(0)).

This provides a general method for integrating a function *f* when f^{-1} can be integrated.

Demonstration of the Method

For example, let $f(x)=a^x$ where a is a constant.

a^x can equally well be written as $e^{x \ln a}$, which can be dealt with by other integration methods anyway, but the example will serve to outline the method.

$f^{-1}(y)$ is $\log_a y$, which equals (ln y)/(ln a).

So $\int f^{-1}(y)=\frac{y\ln y-y}{\ln a}+c$ (by linearity rule, and ln x integrates as x ln x - x)

Hence $\int_{f(0)}^{f(z)} f^{-1}(y)dy=\frac{f(z)\ln f(z)-f(z)}{\ln a}-\frac{f(0)\ln f(0)-f(0)}{\ln a}$

As f(x) has been defined as a^x,

$$\int_{a^0}^{a^z} f^{-1}(y)dy=\frac{a^z\ln(a^z)-a^z}{\ln a}-\frac{a^0\ln(a^0)-a^0}{\ln a}$$

$$=\frac{za^z\ln a-a^z}{\ln a}-\frac{1\ln 1-1}{\ln a}$$

(Warning! We have assumed that a≠0, as if a=0 then a^0 is undefined – and so, in any case, is ln a, so talking about $\log_a y$ is meaningless. Hence the result will have to specifically exclude the case a=0)

$$\int_{a^0}^{a^z} f^{-1}(y)dy=za^z-\frac{a^z}{\ln a}-\frac{0-1}{\ln a}$$

$$=za^z+\frac{1-a^z}{\ln a}$$

From the Method,

$$\int_0^z a^x dx+\int_{a^0}^{a^z}\log_a y\, dy=z\times a^z$$

$$\Rightarrow\int_0^z a^x dx=za^z-(za^z+\frac{1-a^z}{\ln a})$$

$$=za^z-za^z-\frac{1-a^z}{\ln a}$$

$$=\frac{a^z-1}{\ln a}$$

$$=\frac{a^z}{\ln a}+\frac{-1}{\ln a}$$

As -1/ln a is a constant, and

$$\int_u^v f(x)\,dx \equiv \int_0^v f(x)\,dx - \int_0^u f(x)\,dx$$

hence

$$\int_u^v a^x\,dx = \frac{a^v}{\ln a} + \frac{-1}{\ln a} - \left(\frac{a^u}{\ln a} + \frac{-1}{\ln a}\right)$$

$$= \frac{a^v - a^u}{\ln a} + \frac{(-1)-(-1)}{\ln a}$$

$$= \frac{a^v - a^u}{\ln a}$$

(the constant terms cancelling)

The *only way* that the following congruence can *always* be true (the above shows that it indeed is)

$$\int_u^v a^x\,dx \equiv \frac{a^v - a^u}{\ln a}$$

is if the following is true;

$$\int a^x\,dx = \frac{a^x}{\ln a} + k$$

(for an unknown constant *k*)

where a≠0.

For the special case where $a=0$, $y=0^x$. For $x\neq0$, $y=0$, so the integral of 0^x across any bounds which do not cross $x=0$ is zero (plus *k* of course!).
In the special case where $x=0$ is crossed, the limit of the integral-so-far as x tends to 0 is 0, so we can state (eventually) that

$$\int 0^x\,dx = 0 + k$$

(I do not have the full proof of this, but it should be clear anyway)
However,

$$\lim_{a\to0} \frac{1}{\ln a} = 0$$

(since ln a tends towards negative infinity, hence 1/ln a tends towards zero)

Therefore, applying the earlier result

$$\int a^x\,dx = \frac{a^x}{\ln a} + k$$

$$\lim_{a\to0} \int a^x\,dx = 0 + k$$

which we can simply write as

$$\int 0^x\,dx = 0 + k$$

- which is the answer we already got.

So in general we can state that

$$\int a^x\,dx = \frac{a^x}{\ln a} + k$$

for all $a, x \in \mathbb{R}$.

Obviously other integration techniques could have been used in this instance, but sometimes this method is easier than the alternatives.