Algebra and Geometry
in the
Sixteenth and Seventeenth Centuries

Introduction

After outlining the state of algebra and geometry at the beginning of the sixteenth century, we move to discuss the advances in these fields between 1500 and 1640. A separate section is devoted to the development and use of algebraic geometry by Descartes, Fermat and Newton. We close with an attempt to assess the relative importance of these developments.

State of the Arts: Chuquet and Pacioli

At the beginning of the sixteenth century, mathematics was dominated by its Greek heritage and therefore by the study of geometry. But algebra was not wholly absent, and significant advances in notation had been made towards the end of the fifteenth century. Two works were particularly important in this regard: Nicolas Chuquet's (c.1440-c.1488) *Triparty* (1484) and Luca Pacioli's (c.1445-1517) *Summa* (1494). Pacioli's symbolism was limited, consisting mostly of abbreviations. Although Chuquet's symbolism was more advanced, the influence of this work was limited by its small circulation: it was not properly published until 1880.

Algebraic Advances: Cardano, Bombelli, Viète and Harriot

The crucial advances on cubic equations were made in the secretive climate of sixteenth century Italian academia. The story of their solution is complicated and involves many interesting biographical details.[1] Let it here suffice to say that Girolamo Cardano (1501-76) and his student Ludovico Ferrari (1522-65) were together able to solve not only all thirteen types of cubic equation, but all twenty types of quartic equation. The results were published in Cardano's *Ars Magna* (1545). Remarkably, these advances were made using the old

[1] The chief players in this story were Scipione del Ferro (1465-1526), Antonio Maria Fior, Niccolò Tartaglia (1499-1557), Girolamo Cardano (1501-76) and Ludovico Ferrari (1522-65).

rhetorical algebra, in which, for example, $x^3 + cx = d$ was written as *cube and things equal to numbers*.

Cardano accepted the negative solutions to which his rules sometimes led, though he described them as "fictitious". He struggled, however, when his rules required him to find the square root of a negative number, stating that such numbers as $\sqrt{-15}$ are as "refined as they are useless".

Some of these difficulties were later ameliorated by Rafael Bombelli (1526-72) whose *Algebra* (1572) included the first discussion of what we now call 'complex numbers'.[2] Bombelli also developed an algebraic notation similar to that of Chuquet.

Although his algebra was syncopated rather than symbolic, further notational advances were made by François Viète (1540-1603). His *Isagoge* (1591) used letters not only for unknowns but also for general coefficients.

Another who did important work on cubics and quartics was Thomas Harriot (1560-1621). Expressed in modern notation, Harriot was first to see that the three solutions of $(x - a)(x - b)(x - c) = 0$ are $x = a$, $x = b$ and $x = c$.[3]

Progress in Geometry: Harriot, Roberval, Desargues and Pascal

Harriot also achieved the rectification and quadrature of the equiangular spiral and in about 1630, Gilles Personne de Roberval (1602-75) showed that the area under one arch of the cycloid is three times that of the generating circle.

Probably the most significant development in geometry over the period in question was that of Girard Desargues' (1591-1661) projective geometry. Projective geometry is concerned with geometrical properties that are invariant under projection. Desargues' insight, presented in his *Brouillon Project* (1639), was that viewed from the vertex of a cone, all sections of that

[2] See **SB** 8.A5(b), p. 264.
[3] See **SB** 9.D3(a), p. 292.

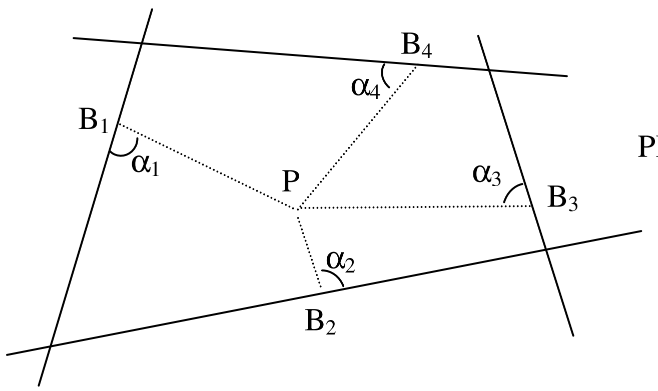
cone appear the same; they all appear as a circle. All conics are therefore projections of the circle. Among other things, this means that many theorems about circles can be interpreted as theorems about other conics, and visa-versa.

Blaise Pascal (1623-62) used Desargues' methods to prove what is now known as Pascal's theorem. The theorem states that if a hexagon is inscribed within a conic, then the 'meets' of lines produced from opposite sides of the hexagon lie on a straight line.

Algebraic Geometry: Descartes, Fermat and Newton

René Descartes'' (1596-1650) *La Géométrie*, one of three appendices to his *Discours de la Méthode* (1637), introduced the world to algebraic geometry. In effect, the method was to set up a system of axes, chosen according to the problem in hand, relative to which the position of any point in the plane may be specified by two coordinates, *x* and *y*, thus allowing the problem to be investigated algebraically.

The power of Descartes' method is illustrated by his treatment of Pappus' problem. That problem involves finding the locus of points P such that, given four lines in a plane, the product of P's distances from two of the lines stands in a specified ratio to the product of P's distances from the other two lines, where those distances are measured at given angles to the lines.



$$PB_1 \cdot PB_2 = k \cdot PB_3 \cdot PB_4 \quad (1)$$

Pappus had stated that the locus of P was a conic section, a result that Descartes was able to prove. Algebraic analysis of Pappus' problem leads to expressions for the distances between

P and the given lines which, substituted into equation (1) above, yield a second degree equation specifying the locus of P in terms of the relationship between x and y. That equation allows us to find values of y corresponding to any given value of x, and thus to plot the curve.

Descartes also applied the method to the equivalent problem involving five, six or more lines. In these cases the equation which represents the locus could have a degree of greater than two.[4]

A similar system was developed independently by Pierre de Fermat (1601-65). Fermat stated that equations in two unknowns represent curves in the plane, and used a system of rectangular axes. Although the work was not published in his lifetime, many of the ideas were formulated *before* the appearance of Descartes' *La Géométrie*.[5] Indeed, while this branch of mathematics (Cartesian geometry) bears Descartes' name, "Fermat's approach is much closer to the modern treatment of the subject".[6]

Isaac Newton (1642-1727) used Descartes' system in his study of cubics in the 1660's. Newton showed that as many as 72 different types of curve could be represented by cubic equations (later mathematicians added another six). Although seemingly unaware of Desargues' work, Newton went on to claim that every cubic could be seen as the shadow (i.e. projection) of one of just five different cubics.

The Significance of these Developments

While Desargues' projective geometry was put to good use by Pascal and others, these methods were not widely used until their rediscovery in the nineteenth century.[7] And if the sixteenth century began with the dominance of geometry, then by the end of the seventeenth century algebra was in the ascendancy.

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[4] MA 290 *Unit 8: Descartes: Algebra and Geometry*, p. 16.
[5] Stuart Hollingdale, *Makers of Mathematics* (London: Penguin, 1994), p. 138.
[6] Stuart Hollingdale, *Makers of Mathematics* (London: Penguin, 1994), p. 139.
[7] Stuart Hollingdale, *Makers of Mathematics* (London: Penguin, 1994), p. 159.

Viète and his contemporaries had regarded algebra as a method of analysis which should be supplemented by synthesis using the classical geometric methods.[8] Even the formulas for solving cubic and quartic equations were originally proved by geometric means.[9]

However, the study of these equations not only showed an increased proficiency in algebra, it also marked a move away from geometric thinking; unlike x, x^2 and x^3, x^4 has no straightforward geometric interpretation. Moreover, it was these researches that first forced mathematicians to think about complex numbers.

While algebra was already making inroads into the dominance of geometry, it was the power of Descartes' algebraic methods, especially applied to higher plane curves (those other than straight lines and conics), that led mathematicians to both use and trust algebra. Mathematicians came to think that algebraic analysis need not be supplemented by geometric synthesis. The methods had not only created new ties between algebra and geometry, they had encouraged mathematicians to grant algebra an authority of its own.

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[8] MA290 *Unit 8: Descartes: Algebra and Geometry*, pp. 3-6 and 19.

[9] J.J. O'Connor and E.F. Robertson, "Quadratic, cubic and quartic equations" *MacTutor History of Mathematics Archive* <www-history.mcs.st-andrews.ac.uk/history/HistTopics/Quadratic_etc_equations.html> Accessed 23 May 2003