

Was there a Revolution in Geometry in the Nineteenth Century?

Introduction

Arguing that there was a revolution in nineteenth century geometry, I naturally concentrate on the area of mathematics in which that revolution occurred. Therefore, after defining the term ‘revolution’, I only *briefly* summarise the non-revolutionary geometrical developments of the period. The revolutionary developments concern the origin of non-Euclidean geometry, and the history of this topic is traced in more detail. After explaining these developments and showing that they fit our definition of ‘revolution’, I go on to explain some of the repercussions of this revolution in other areas of mathematics.

On the Definition of “Revolution”

In general, the term ‘revolution’ means, roughly, *the overturning of the establishment or of the established system*. Tailoring this to our subject matter, we might define a ‘mathematical revolution’ as follows: *the rejection of assumptions which were previously deeply entrenched and were central to our understanding of an important branch of mathematics*.

Non-Revolutionary Developments

The most important of the non-revolutionary developments in nineteenth century geometry was the rebirth of projective geometry. Jean Victor Poncelet (1788-1867) and Joseph Gergonne (1771-1859) introduced *the principle of duality*, which allows theorems about ‘lines’ and ‘points’ to be reinterpreted as new theorems with these two terms interchanged. Problems with this principle were tidied up by Julius Plücker (1801-68) who, along with August Ferdinand Möbius (1790-1868), also pioneered the use of algebraic methods in the study of projective geometry. In the meantime Jacob Steiner (1796-1863) championed the classical tradition, producing new results in synthetic geometry. While all this work

undoubtedly added to geometry, it did not disturb any of the assumptions that shaped the understanding of existing geometrical notions; it was not revolutionary.

The Revolution of non-Euclidean Geometry

Non-Euclidean geometry stemmed from a dispute over the *parallel postulate*, which Euclid (c. 300 BC) had used in his *Elements*. The postulate in question states that

If a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than two right angles. (SB 3.B1, p. 101)

Together with his first four postulates and five common notions, this, Euclid's fifth postulate, characterised what we now refer to as 'Euclidean geometry'. However, this postulate lacked the intuitive plausibility of the others, which by contrast seemed self-evident. Nevertheless, disputants did not doubt the *truth* of the postulate, but questioned only its status *as a postulate*. In his *A Commentary on the First Book of Euclid's Elements*, Proclus (411-85) wrote that it "ought to be struck from the postulates altogether ... it is a theorem".¹ In the same work Proclus attempted to reduce the postulate to theorem by proving it from the other assumptions. However, he implicitly assumed that parallel lines are a constant distance apart. He had simply substituted Euclid's postulate for an equivalent postulate of his own.

Similar stories could be told of many attempted proofs, the best of which succeeded only in establishing the equivalence of the parallel postulate and some other claim. Among these equivalent claims are the following four:

- The angle sum of a triangle is two right angles. (Nasir Eddin al-Tusi, 1201-74)
- Given any triangle and any line segment, there exists a second triangle similar to the first and having that segment as its base. (John Wallis, 1616-1703)
- Any line through any point interior to any given angle will meet (at least) one side of that angle. (Adrien Marie Legendre, 1752-1833)

¹ SB 3.B1(b) *On Postulate 5*, p. 104.

- Given a line and a point not on that line, there is exactly one line through that point which is parallel to the given line.²

Gerolamo Saccheri (1677-1733) and Johann Hienrich Lambert (1728-1777) also worked on the problem of parallels. In his *Euclid Freed of Every Flaw* (1733), Saccheri assumed the opposite of the parallel postulate and attempted to derive a contradiction. Although he showed that the angle, α , in a ‘Saccheri quadrilateral’ (see below) could not be obtuse, his ‘refutation’ of the *hypothesis of the acute angle* (HAA) was unsound. He therefore failed to establish the Euclidean position, according to which α is a right angle.



Although Lambert took a similar approach in the 1760s, he did not believe that he had proved the parallel postulate and left it open whether such a proof was possible. Among other things, Lambert showed that in a geometry based on the HAA the area of a triangle would be proportional to its *angular defect*, to the difference between π and the sum of its angles (measured in radians).

While no one had been able to prove the parallel postulate, neither had anyone yet seriously believed that a non-Euclidean geometry could be consistent. Indeed, the Euclidean system had enjoyed the commitment of the entire mathematical community for over 2000 years. To doubt the parallel postulate was ‘unthinkable’.³

But in the early nineteenth century, things began to change. In 1818 F.K. Schweikart (1780-1859) wrote a letter to Carl Friedrich Gauss (1777-1855) revealing that he was

² This last alternative postulate is generally known as “Playfair’s Axiom” after John Playfair (1748-1819), who used it in his popular *Elements of Geometry* (1795).

³ MA290 *Unit 13: Non-Euclidean Geometry*, p. 13.

exploring an ‘astral geometry’, in which the angle sum of a triangle is less than two right angles. It seems clear that he believed this geometry to be perfectly consistent.⁴

The first systematic accounts of non-Euclidean geometry were supplied by Nicolai Lobachevskii (1792-1856) and Janos Bolyai (1802-60). Although the two men reached their ideas independently, their work is remarkably similar and their first publications were almost simultaneous: Lobachevskii’s appeared in 1829 and Bolyai’s as an appendix to his father’s geometrical treatise also dated 1829, but not published until 1831 or 1832.⁵ Both men derived various algebraic formulae applicable in this new, non-Euclidean geometry. According to one such formula – known to Schweikart’s nephew F.A. Taurinus (1794-1874) and seemingly anticipated by Lambert – non-Euclidean geometry corresponds to that on the surface of a sphere of imaginary radius.

Lobachevskii and Bolyai were relatively unknown mathematicians, producing mathematics that few could understand, and publishing their works in obscure places. Understandably, their now seminal contributions to mathematics were almost completely ignored. Two events conspired to change things. Firstly, after his death in 1855 the publication of Gauss’ papers and correspondence showed the world that at least one first-rate mathematician thought the possibility of a non-Euclidean geometry worth exploring. Gauss thought highly of Lobachevskii’s work and had exchanged letters with the Bolyais and others, stating that their conclusions agreed with his own unpublished researches.

The second event was the construction, by Eugenio Beltrami (1835-1900), Felix Klein (1849-1925) and Henri Poincaré (1854-1912), of several *models* for non-Euclidean geometry. One of Beltrami’s models, devised in 1868, was the *pseudosphere*: a surface of revolution generated by spinning the tractrix about its asymptote. Since these models for non-Euclidean geometry exist *within* Euclidean geometry, any contradiction in the new geometry would

⁴ MA290 *Unit 13*, p. 13; **SB** 16.B1, pp. 522-3.

⁵ Sources seem to disagree on the actual publication date, perhaps because Wolfgang Bolyai’s *Tentamen* was as two-volume work.

show up as a contradiction in the old, Euclidean geometry. As such, the existence of these models clearly showed non-Euclidean geometry to be as consistent as Euclidean geometry.

G.F.B. Riemann (1826-66) produced two further striking geometrical results. Firstly, if others of Euclid's axioms are reinterpreted, a consistent geometry *can* be based upon the previously rejected *hypothesis of the obtuse angle*. The resulting geometry applies, for example, on the surface of a sphere. In this system the area of a triangle is proportional to its *angular excess*. Secondly, Riemann extended these ideas to reveal that a consistent geometry can be developed on any given surface, an idea leading to infinitely many new geometries.

Consider the following illustrations of just how radically different were the new geometries from the familiar Euclidean geometry. In Euclidean geometry: (i) the angle sum of a triangle is two right angles, (ii) two lines parallel to a third line are parallel to each other, (iii) two lines cannot enclose a space, (iv) given a line and a point not on it, there is one line through that point parallel to the given line, and (v) there is no absolute unit of length. In Lobachevskii and Bolyai's non-Euclidean geometry: the angle sum of a triangle is less than two right angles; two lines parallel to a third may intersect; given a line and a point not on it, there are many lines through that point which are parallel to the given line; and there is an absolute unit of length. (Lambert and Schweikart had noticed this last result.) In Riemann's spherical geometry: the angle sum of a triangle is greater than two right angles, parallel lines do not exist, two lines do enclose a space, and again there is a unit of absolute length.

The non-negotiable nature of claims like (i)-(v) had not merely been central to our understanding of geometry, it was *partially constitutive* of that understanding. Indeed, this was perhaps the main reason why the possibility of non-Euclidean geometries was so hard to accept. With the discovery of non-Euclidean systems, the status of those claims changed. They no longer characterized geometry itself, but only one of many possible geometries.

Referring this back to our definition, it seems that non-Euclidean geometry was truly revolutionary.

Repercussions

The realisation that Euclid's system was not unique led mathematicians to analyse his work more carefully. Close scrutiny of his arguments showed that Euclid's postulates and common notions were not sufficient to ground Euclidean geometry; many of his propositions depended upon assumptions that had not been made explicit. This prompted mathematicians to 're-axiomatise' Euclidean geometry. The definitive re-axiomatisation was given by David Hilbert (1862-1943), who in his *Foundations of Geometry* (1899) replaced Euclid's postulates and common notions with 21 axioms of his own.

Axiomatisation became *de rigueur*, spreading to other areas of mathematics, including arithmetic and algebra. But the experience with geometry emboldened mathematicians to play with various sets of axioms, each defining their own geometry, algebra or arithmetic. Any set of axioms was now permissible, so long as it was consistent. The discovery of non-Euclidean geometry had triggered the so-called 'liberation' of algebra and geometry.

With so many geometries in play, it also became natural to ask whether these different systems have anything in common. In the 1870s Felix Klein revealed the underlying unity in the obvious diversity when he proposed that each geometry be understood as the study of the properties of figures that are invariant under a particular group of transformations. By examination of the groups in question he showed other geometries to be special cases of projective geometry.

The proliferation of axiomatic systems also had consequences in the philosophy of mathematics, leading mathematicians and philosophers to ask what mathematics is *about*. Mathematics had previously been regarded a logical science revealing, *a priori*, the necessary structures of reality. But it now seemed easier to say that mathematics is a game in which we

merely deduce the consequences of whatever rules we should care to invent. Some rules may be more interesting than others, but we cannot know *a priori* which are more true to reality.

Conclusion

As promised, I began by describing the non-revolutionary developments in nineteenth century geometry. I went on to discuss the origin of non-Euclidean geometry, beginning with the long running dispute over the status of the parallel postulate. The major change occurred when Lobachevskii and Bolyai became the first to publish work developing what they claimed to be a consistent non-Euclidean geometry. But the revolution was not complete until the mathematical world took notice of these results, and this would only occur with the publication of Gauss' correspondence and the construction of models for the non-Euclidean plane. Further changes were wrought by Riemann, who showed how to develop other consistent geometries on an infinite variety of curved surfaces. These revolutionary developments had consequences in other areas: the re-axiomatisation of Euclidean geometry, the proliferation of axiomatic systems of geometry and algebra, the subsequent reunification of geometry, and the reconceptualisation of mathematics as having no necessary relation to reality. In fact, not only was non-Euclidean geometry revolutionary in itself, these repercussions were so momentous that one can barely understate the importance non-Euclidean geometry in the history of mathematics.