

Fourier methods in option pricing

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1. Introduction

This article is a short introduction into Fourier option pricing methods, for both European and Bermudan options. Section 2 considers the pricing of European options using Fourier transform methods. Section 3 considers the application of these techniques to the pricing of Bermudan, or early-exercise options.

2. Pricing European options

There are various ways to price European options via Fourier inversion. Before we consider such methods, it is worth mentioning why Fourier option pricing methods can be useful. First recall that the risk-neutral valuation theorem states that the forward price of a European call option on a single asset S can be written as:

$$C(S(t), K, T) = \mathbb{E}[(S(T) - K)^+] \quad (2.1)$$

where C denotes the value, T the maturity and K the strike price of the call. The expectation is taken under the T -forward probability measure. As (2.1) is an expectation, it can be calculated via numerical integration, provided we know the density in closed-form. For many models the density is either not known in closed-form, or quite cumbersome, whereas its characteristic function is often much easier to obtain. A good example hereof in finance is the Variance Gamma model, introduced by Madan and Seneta [1990]. Its density involves a Bessel function of the third kind, whereas its characteristic function only consists of elementary functions.

Heston [1993] was among the first to utilise Fourier methods to price European options within his stochastic volatility model. Since Heston's seminal paper, the pricing of European options by means of Fourier inversion has become more and more commonplace. Heston's approach starts from the realisation that (2.1) can be recast as:

$$C(S(t), K, T) = F(t, T) \cdot \mathbb{S}(S(T) > K) - K \cdot \mathbb{P}(S(T) > K) \quad (2.2)$$

with $F(t, T)$ the forward price of the asset and \mathbb{P} and \mathbb{S} respectively the T -forward probability measure and the measure induced by taking the asset price as the numeraire.

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The cumulative probabilities in (2.2) can subsequently be found by Fourier inversion, an approach dating back to Lévy [1925], Gurland [1948] and Gil-Pelaez [1951]. As this approach necessitates the evaluation of two Fourier inversions, and is inaccurate for out-of-the-money options, due to cancellation, we do not discuss it here in further detail. Instead we focus on more recent approaches, due to Carr and Madan [1999], and Raible [2000] and Lewis [2001].

Carr and Madan's approach was to consider the Fourier transform of the damped European call price, with respect to the logarithm of the strike price:

$$\psi(v, \alpha) \equiv \int_{-\infty}^{\infty} e^{ivk} e^{\alpha k} C(k) dk = \frac{\phi(v - i(\alpha + 1))}{-(v - i\alpha)(v - i(\alpha + 1))} \quad (2.3)$$

and subsequently invert this to arrive at the desired call price:

$$C(S(t), K, T) = \frac{e^{-\alpha k}}{2\pi} \int_{-\infty}^{\infty} e^{-ivk} \psi(v, \alpha) dv = \frac{e^{-\alpha k}}{\pi} \int_0^{\infty} \text{Re}(e^{-ivk} \psi(v, \alpha)) dv \quad (2.4)$$

In (2.3) - (2.4) $\phi(u)$ is the characteristic function, $\phi(u) = E[\exp(iu \ln S(T))]$. Sufficient conditions for (2.4) to exist are that the damping factor $\alpha > 1$, and that the $(\alpha+1)^{\text{st}}$ moment of the asset, $\phi(-(\alpha+1)i)$, is finite. The first condition is required to make the damped call price an L^1 -integrable function, which is a sufficient condition for the existence of its Fourier transform.

Whereas Carr and Madan took the Fourier transform with respect to the strike price of the call option, Raible [2000] and Lewis [2001] used an approach which is slightly more general in that it does not require the existence of a strike in a payoff. Raible took the transform with respect to the log-forward price, Lewis used the log-spot price. Note that for all three methods, the Fourier transform of the option price can be decoupled into two parts, a payoff-dependent part, the payoff transform, and a model-dependent part, the characteristic function.

One of the restrictions on the damping factor for a call price is that it must be larger than one. However, as Lewis [2001] and Lee [2004] point out, this is not a real restriction if we recast (2.4) as a contour integral in the complex plane. Shifting the contour, equivalent to varying α in (2.4), and carefully applying Cauchy's residue theorem leads to the following option pricing equation:

$$C(S(t), K, T, \alpha) = R(F(t, T), K, \alpha) + \frac{e^{-\alpha k}}{\pi} \int_0^{\infty} \text{Re}(e^{-ivk} \psi(v, \alpha)) dv \quad (2.5)$$

where the residue term $R(F, K, \alpha)$ equals 0 for $\alpha > 0$, $\frac{1}{2}F$ for $\alpha = 0$, F for $\alpha \in (-1, 0)$, $F - \frac{1}{2}K$ for $\alpha = -1$, and $F - K$ for $\alpha < -1$. For values of $\alpha < -1$ for example this means that the integral in (2.5) yield the value of a put, from which we obtain the price of a call via put-call parity.

As far as the numerical implementation of (2.5) goes, the appropriate numerical algorithm depends as always on the user's demands. If one is interested in the option price at a great many strike prices, (2.5) can be discretised in such a way that it is

amenable to use of the Fast Fourier Transform (FFT), as detailed in Carr and Madan [1999] and section 12.8. If one is calibrating a model to a volatility surface, one often only needs to evaluate option prices at a handful of strikes, at which point a direct integration of (2.5) becomes computationally advantageous. Important points to consider in approximating the semi-infinite integral in (2.5) are the discretisation error, the truncation error and the choice of contour or damping factor α . Lee [2004] extensively analyses these choices when (2.5) is discretised using the Discrete Fourier Transform (DFT), and proposes a minimisation algorithm to determine the parameters of the discretisation.

Lord and Kahl [2007] propose a different approach. If an appropriate transformation function is available which maps the semi-infinite interval into a finite one, the truncation error can be avoided altogether. This leaves the discretisation error, which can be controlled by using adaptive integration methods. Finally, the speed and accuracy of the integration algorithm can be controlled by choosing an appropriate value of α . A good proxy for the optimal value of α , that value which minimises the approximation error given a fixed computational budget, is:

$$\alpha^* = \arg \min_{\alpha \in (\alpha_{\min}, \alpha_{\max})} \left| e^{-\alpha k} \psi(0, \alpha) \right| = \arg \min_{\alpha \in (\alpha_{\min}, \alpha_{\max})} -\alpha k + \frac{1}{2} \ln(\psi(0, \alpha)^2) \quad (2.6)$$

where $(\alpha_{\min}, \alpha_{\max})$ is the allowed range of α , corresponding to $\phi(-(\alpha+1)i) < \infty$. This choice of contour is closely linked to how the optimal contour is chosen in saddlepoint approximations. That α really can have a significant impact on the accuracy should become clear from the following example.

Example 2.1 – Impact of α on the numerical approximation in Heston’s model

As an example we look at the impact of α in Heston’s stochastic volatility model, see section 8.18. The parameters we pick are $\kappa = \omega = 1$, $\rho = -0.7$, $\theta = v(0) = 0.1$, $F = 1$, $K = 1.5$ and the time to expiry is 0.1 years. Figure 1 shows the impact of α on the approximation error when using two different ways to discretise (2.5).

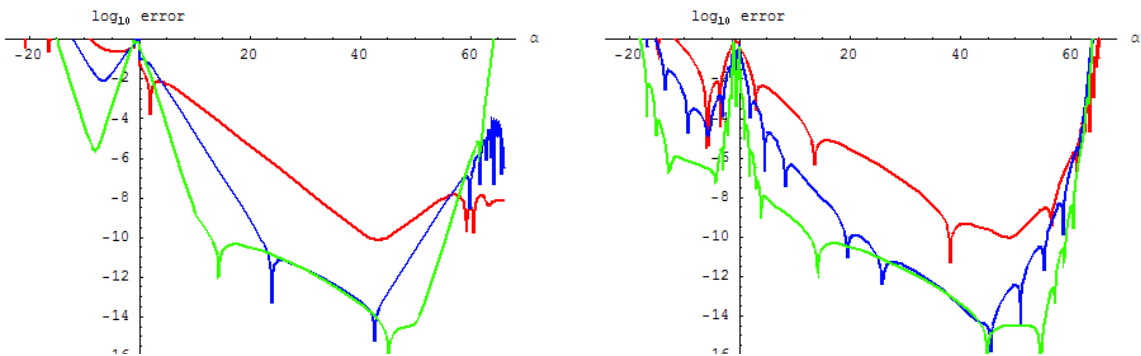


Figure 2.1: Impact of α using Lee’s [2004] DFT discretisation, or Gauss-Legendre quadratures (right). The various colours represent the no. of abscissae used (8, 16 or 32).

If one plots the function which is minimised in (2.6), one obtains a very similar pattern, suggesting that α^* is indeed close to optimal.

Finally, Andersen and Andreasen [2002] and Cont and Tankov [2004] have suggested that the Black-Scholes could be used as a control variate in the evaluation of (2.5), i.e. we could subtract the Black-Scholes integrand from the integrand, and subsequently add the Black-Scholes price back to the equation. While this could work for some models, the approach does require a good correspondence between both characteristic functions, and also requires an educated guess as to which Black-Scholes volatility should be used.

3. Pricing Bermudan and American options

Now we can price European options using Fourier methods, the next question is whether options with early-exercise features can be priced in a similar framework. The answer is affirmative. The first paper to attempt this in the framework of Carr and Madan was O'Sullivan [2005], who extended the QUAD method of Andricopoulos, Widdicks, Duck and Newton [2003] to allow for models where the density is not known in closed-form, but has to be approximated via Fourier inversion. This method is $O(MN^2)$, where M is the number of time steps and N is the number of discretisation points used in a one-dimensional model.

Building upon a presentation by Reiner [2001], Lord, Fang, Bervoets and Oosterlee [2008] noticed that the key to extending Carr and Madan's approach to early-exercise options, was to abandon the idea of working with an analytical Fourier transform of the option payoff, but to numerically approximate it. If at time t_m we have an expression for the value of the option contract, then its continuation value at t_{m-1} can be obtained by calculating its convolution with the transition density. As we know the Fourier transform of a convolution is the product of the individual Fourier transforms, all we need to do is numerically calculate the Fourier transform of the continuation value. Having calculated the continuation value, we obtain the value at time t_{m-1} simply by comparing to the exercise value. The CONV method of Lord et al. [2008] utilises the FFT to approximate the convolutions. As such, the algorithm is $O(MN \log N)$. For Bermudan options the algorithm is certainly competitive with the fastest partial-integro differential (PIDE) methods, see the numerical comparison in Lord et al. [2008]. The prices of American options can be obtained via Richardson extrapolation. It is here that PIDE methods have an advantage. Another area where PIDE methods are advantageous is at the choice of gridpoints – as the CONV method employs the FFT, the grid for the logarithm of the asset price needs to be uniform. This makes it harder to place discontinuities on the grid, something which is much easier to achieve in e.g. the QUAD method or PIDE methods. Extensions of the CONV method to multiple dimensions can be found in Leentvaar and Oosterlee [2008].

Finally, we mention a recent paper by Fang and Oosterlee [2008], in which Bermudan options are efficiently priced via Fourier-cosine series expansions. While this method is also $O(MN \log N)$ and has some similarities with the CONV method, a great advantage is that the exercise boundary is directly solved, as in the QUAD method. Hence, the cosine series coefficients can be calculated exactly, instead of being approximated, which is the case in the CONV method. Whereas the convergence of the CONV method is dictated by the chosen Newton-Cotes rule, the convergence of the COS method is dictated by the rate of decay of the characteristic function.

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