A motivation for conditional moment matching

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This version: February 28th, 2005
First version: February 22nd, 2005

ABSTRACT

One can find approaches galore in the literature for the valuation of Asian basket options. When the number of underlyings is large one has to resort to bounds or approximations to value these options. In this respect, Curran [1994] and Rogers and Shi [1995] very successfully applied a conditioning approach. Recently, Lord [2005] combined their approach with the traditional ad-hoc moment matching approaches, to obtain an approximation which is extremely accurate and has an analytical bound on its error. Here we review this approach and extend the results to multiple conditioning variables, along the lines of Vanmaele, Deelstra and Liinev [2004].

Keywords: Asian option, average price option, basket option, conditioning approach, moment matching, lower bound, upper bound, analytical approximation.

The author is grateful to the organisers of the 3rd Actuarial and Financial Mathematics Day in Brussels for providing him the opportunity to speak at their conference.

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1. Introduction

This paper deals with the pricing of European options on arithmetic averages. If the average is a time average of a single underlying asset, these options are referred to as Asian options. Another possibility is that the average is taken over several assets at the same time instant; these options are referred to as basket options. Of course, mixtures of Asian and basket options exist in the market. For instance, the average could be taken over time and over several assets, creating what we will refer to as an Asian basket option.

The reason for the existence of such options is clear. As far as basket options are concerned, large companies may want to buy some downside protection on their investments. One possibility to achieve this would be to buy an option on a basket that is representative for the investments of the firm. What about Asian options, i.e. options on a time average of a single underlying? A pure European option on this asset would exhibit a large dependence on the final value of the underlying asset, and as such the option is quite sensitive to large shocks or price manipulation. To avoid such issues, many financial contracts often contain a so-called "Asian tail", which means that the final payoff is based on the average price of the underlying over a time interval before the expiry date. Recently Schrager and Pelsser [2004] have shown that unit-linked guarantees contain rate of return guarantees, which closely resemble Asian options. Given the fact that fair value calculations are currently the talk of the town, it is highly important to be able to value these types of contracts.

In the Black-Scholes model it is already not straightforward to price these types of contracts, the main reason for this being that no closed-form probability law exists for the sum of correlated lognormal random variables. As the valuation of these options is already mathematically interesting in the Black-Scholes model, many papers, including ours, are based in this lognormal setting. Within this model many an approach has been used to value or approximate these options. Broadly speaking we can divide these methods into five classes: approaches based on Monte Carlo simulation, the numerical solution of partial differential equations (PDEs), integral transforms, analytical approximations or analytical bounds. We will not attempt to give an overview of all these approaches here, we refer the interested reader to Lord [2005] and references therein.

In principle, the most flexible approach when considering multiple underlyings is probably a Monte Carlo simulation. Aside from the fact that it is very easy to implement, a large advantage is that we can easily allow for more realistic dynamics in the model. However, even though excellent control variates exist within the Black-Scholes model, the method can still be somewhat computationally intensive, not to mention the additional problem of computing sensitivities with respect to model and market parameters. Nowadays however, many structured products with a basket as their underlying, use caps and floors on the performance of individual underlyings, so that Monte Carlo simulation or the PDE approach is the only method that can be used. Here we ignore these additional features, and consider a simple European arithmetic Asian basket option.

Since financial institutions demand quick and accurate answers for the value of a derivative and its Greeks, large parts of the literature have focused on analytical approximations and bounds. Probably one of the most widely known and used approximations is that of Levy [1992], who approximates the arithmetic average with a lognormal random variable, such that the first two moments coincide with that of the true distribution. A problem that this approximation shares with other ad-hoc moment matching approaches, is that the size of their error is not known analytically. Furthermore, most of these approximations only tend to work well for low to moderate volatility environments. The first shortcoming has certainly motivated researchers to come up with sharp lower and upper bounds on the value of these options. A seminal paper in this area is that of Rogers and Shi [1995]. In the context of Asian options, they derived a sharp lower bound and an upper bound on the value of an Asian option. The technique used to derive the lower bound is remarkably simple, but very effective – they condition on a variable that is highly correlated with the basket, and then apply Jensen's inequality to find a very sharp lower bound. Curran [1994] arrived at the same lower bound, and was the first to observe that the payoff of Asian basket options can be split into two parts: one that can be calculated exactly, and one that has to be approximated. This approach yielded Curran's so-called "sophisticated" approximation. Recently, Lord [2005] showed that this approximation of Curran actually diverges when the strike price tends to infinity. Combining the ideas of Rogers and Shi and Curran, he introduces the class of partially exact and bounded (PEB) approximations, which are guaranteed to lie between Rogers and Shi's lower bound, and a sharpened version of Rogers and Shi's upper bound (due to Nielsen and Sandmann [2003] and Vanmaele, Deelstra, Liinev, Dhaene and Goovaerts [2005]).

In this paper we will, in the next section, first review the conditioning approaches of Rogers and Shi and Curran. In Vanmaele, Deelstra and Liinev [2004] it is shown how to extend the lower bound of Rogers and Shi so that we can condition on two random variables. Here we trivially extend this lower bound, as well as the sharpened upper bound of Rogers and Shi, to allow for an arbitrary number of conditioning variables. In the third section we review the results of Lord [2005], which provide a clear motivation for conditional moment matching. Building on the results of the second section, we can extend the PEB approximations to allow for multiple conditioning variables. Finally, we show that a recent bounded approximation of Vanmaele, Deelstra and Liinev [2004], which also matches the first two conditional moments, satisfies all requirements of a PEB approximation. We end thepaper with a brief numerical illustration and some conclusions and recommendations. The focus throughout the paper will not be on exact expressions required to calculate the various bounds and approximations, but on the rationale behind the approaches.

Before starting the next section, we will first introduce some notation. As mentioned, we will base ourselves in the Black-Scholes framework. For notational convenience we will work with a constant parameter Black-Scholes model, although all results still hold when these parameters are deterministic functions of time, and the growth and spot rate are Gaussian. We assume the underlying assets S_i, i = 1, …, N and the money market account M evolve according to the following stochastic differential equation (SDE):

$$\begin{aligned} \frac{dS_i(t)}{S_i(t)} &= \mu_i dt + \sigma_i dW_i(t) \\ \frac{dM(t)}{M(t)} &= r dt \end{aligned} \tag{1.1}$$

where all Brownian motions are correlated with instantaneous correlation matrix R. Throughout the document we will assume, without loss of generality, that the current date is 0. The underlyings of all options we consider in this paper will be an Asian basket, which at the maturity date T will be defined as B(T) in:

$$\begin{aligned} B(T) &= \sum\nolimits_{i=1}^{N} w_i A_i(T) \\ A_i(T) &= \int_0^T S_i(t)\rho_i(t)\,dt \end{aligned} \tag{1.2}$$

Here, the weights w_i are positive and sum to 1, and similarly all ρ_i are non-negative functions, integrating to 1 over [0,T]. We will only consider newly issued, non-forward-starting call options on this Asian basket. This is no loss of generality. Put options can be priced via the Asian put-call parity, whereas running average options can be treated as newly issued ones, with a correction to the strike price. Finally, forward-starting options pose no problems when interest rates are deterministic or Gaussian. For ease of exposure we will mostly deal with forward prices in our analysis. The forward price of the Asian basket call option is equal to its expected value under the risk-neutral probability measure $\mathbb{Q}$, conditional upon all information known at time 0:

$$c_B(T,K) = \mathbb{E}_0^{\mathbb{Q}}[(B(T)-K)^+] \tag{1.3}$$

In the remainder we will leave out the superscript indicating the measure and the subscript indicating at which time the expectation is evaluated, unless any confusion can arise. Having introduced the notation we will use, we are now ready to turn to the next section.

2. The conditioning approaches

The most successful approximations and bounds all rely heavily on results first derived by Rogers and Shi [1995] and Curran [1994]. We briefly review their approaches here, whereafter we extend them to allow for multiple conditioning variables, something which was done for two conditioning variables by Vanmaele, Deelstra and Liinev [2004]. We will here use Curran's idea of decomposing the Asian option into two parts: one that can be calculated exactly, and one that has to be approximated. Suppose that we have a normally distributed random variable Λ with the convenient property that $\Lambda \geq \lambda(K)$ implies that $B(T) \geq K$. Examples of such random variables will be given shortly. Following Curran, we can then write:

$$\begin{aligned} c_B(T,K) &= \mathbb{E}[(B(T)-K)^+ 1_{[\Lambda<\lambda(K)]} + (B(T)-K)^+ 1_{[\Lambda\geq\lambda(K)]}] \\ &= \mathbb{E}[(B(T)-K)^+ 1_{[\Lambda<\lambda(K)]} + (B(T)-K) 1_{[\Lambda\geq\lambda(K)]}] \\ &\equiv c_1(T,K,\Lambda) + c_2(T,K,\Lambda) \end{aligned} \tag{2.1}$$

As is shown in Curran [1994], it is quite straightforward to calculate the c_2-part, using the convenient property that normally distributed random variables are still normally distributed upon conditioning on a correlated normal random variable. We will therefore refrain from reproducing the exact formulae here. This leaves us with the calculation of the c_1-part, which we can bound or approximate. Let us first however consider several possible random variables Λ, which have the above property. A very natural candidate for such a Λ is the logarithm of the geometric average, which for the Asian basket will be defined as:

$$\begin{aligned} G(T) &= \prod_{i=1}^{N} G_i(T)^{w_i} \\ G_i(T) &= \exp\left(\int_0^T \ln S_i(t)\,\rho_i(t)\,dt\right) \end{aligned} \tag{2.2}$$

An application of the weighted Jensen's inequality shows that $B(T) \geq G(T)$, with equality attained if and only if all components of the average are equal. Defining $\Lambda_{GA} = \ln G(T)$, it is then obvious that when $\Lambda_{GA} \geq \ln K$, we indeed have $B(T) \geq K$. Other possible conditioning variables, see e.g. Vanmaele, Deelstra and Liinev [2004], arise from a first order approximation of the Asian basket $B(T)$ in its driving Brownian motions. In the setting of an Asian basket option, we then obtain the following conditioning variables and their corresponding thresholds:

$$\begin{aligned} \Lambda_{FA1} &= \sum_{i=1}^{N} w_i \int_0^T S_i(0) \exp\left((\mu_i - \tfrac{1}{2}\sigma_i^2)t\right) \cdot \left(1 + \sigma_i W_i(t)\right) \rho_i(t)\, dt \qquad \lambda_{FA1}(K) = K \\ \Lambda_{FA2} &= \sum_{i=1}^{N} w_i \int_0^T S_i(0) \cdot \left(1 + (\mu_i - \tfrac{1}{2}\sigma_i^2)t + \sigma_i W_i(t)\right) \rho_i(t)\, dt \qquad \lambda_{FA2}(K) = K \end{aligned} \tag{2.3}$$

We note that the higher the correlation of Λ with B(T) is, the larger the relative contribution of c_2 to the option price will be. In practice, any of the above conditioning variables is quite highly correlated with B(T), provided that the volatilities of the underlying assets are not too high. This is one of the key points as to why these conditioning approaches work so well – c_2 constitutes a large part of the option price, so that any approximation we make in c_1 will not have a large impact. The larger the volatilities and maturities are, the more important it becomes to have an accurate approximation to c_1.

We now turn to the approximating part c_1. Both Rogers and Shi and Curran used Jensen's inequality to find a lower bound on the value of these options. A lower bound on c_1 simply follows from:

$$\begin{aligned} c_1(T,K,\Lambda) &= \mathbb{E}[\left(B(T)-K\right)^+ 1_{[\Lambda<\lambda(K)]}] = \mathbb{E}[\mathbb{E}[\left(B(T)-K\right)^+ 1_{[\Lambda<\lambda(K)]} \mid \Lambda]] \\ &\geq \mathbb{E}[\mathbb{E}[\left(B(T)-K\right) 1_{[\Lambda<\lambda(K)]} \mid \Lambda]^+] \end{aligned} \tag{2.4}$$

so that then the lower bound becomes the sum of this lower bound and c_2:

$$\begin{aligned} LB(T,K,\Lambda) &= \mathbb{E}[\left(B(T)-K\right)^+ 1_{[\Lambda<\lambda(K)]}] + c_2(t,K,\Lambda) \\ &= \mathbb{E}\left[\left(\mathbb{E}[B(T) \mid \Lambda] - K\right)^+\right] \end{aligned} \tag{2.5}$$

This lower bound can in principle be applied using an arbitrary conditioning variable, not only conditioning variables for which we have the aforementioned property. In Lord [2005] it is shown how to calculate (2.5) in closed-form for an arbitrary conditioning variable, and an arbitrary correlation structure between the various underlyings. This greatly facilitates the computations required for the lower bound, as otherwise we would have to resort to a numerical integration over a discontinuous integrand.

Another approach to approximate c_1 will be pursued in the following section. We will now extend the lower bound so that we can condition on multiple random variables. For two conditioning variables this idea was first pursued in Vanmaele, Deelstra and Liinev [2004], so this is merely a trivial extension of their results. Suppose that we have a conditioning variable Λ and a set of conditioning variables $\mathcal{Z}$, such that for any realisation of the random variables in $\mathcal{Z}$, $\Lambda \geq \lambda(K)$ implies that $B(T) \geq K$. The lower bound on c_1 then becomes:

$$\begin{aligned} c_1(T,K,\Lambda) &= \mathbb{E}[\left(B(T)-K\right)^+ 1_{[\Lambda<\lambda(K)]}] \\ &= \mathbb{E}\left[\mathbb{E}[\left(B(T)-K\right)^+ 1_{[\Lambda<\lambda(K)]} \mid \Lambda, \mathcal{Z}]\right] \\ &\geq \mathbb{E}\left[\mathbb{E}[\left(B(T)-K\right) \mid \Lambda, \mathcal{Z}]^+ \cdot 1_{[\Lambda<\lambda(K)]}\right] \end{aligned} \tag{2.6}$$

so that the resulting lower bound is:

$$LB(T,K,\Lambda,\mathcal{Z}) = \mathbb{E}\left[\mathbb{E}[\left(B(T)-K\right) \mid \Lambda, \mathcal{Z}]^+ \cdot 1_{[\Lambda<\lambda(K)]}\right] + c_2(T,K,\Lambda) \tag{2.7}$$

Note that the first part in (2.7) will typically have to be calculated via a multivariate numerical integration, whereas the second part is the same as before, and can hence be done in closed-form.

Let us now turn to an analysis of the error made by approximating the value of the Asian basket option by the lower bound in (2.7). This upper bound, based on the lower bound, was first derived by Rogers and Shi [1995]. It is based on the following inequality:

$$\begin{aligned} 0 &\le \mathbb{E}[X^+] - \mathbb{E}[X]^+ = \tfrac{1}{2}\left(\mathbb{E}\left[\,|X|\,\right] - |\mathbb{E}[X]|\right) \\ &\le \tfrac{1}{2}\mathbb{E}\left[\,|X - \mathbb{E}[X]|\,\right] \le \tfrac{1}{2}\sqrt{\mathrm{Var}(X)} \end{aligned} \tag{2.8}$$

More recently, Nielsen and Sandmann [2003] and Vanmaele, Deelstra, Liinev, Dhaene and Goovaerts [2005] sharpened this upper bound considerably. We here extend their sharpened version to allow for multiple conditioning variables. Proceeding as above, we find:

$$\begin{aligned} 0 &\le c_B(T,K) - LB(T,K,\Lambda,\mathcal{Z}) \\ &= \mathbb{E}\left[\mathbb{E}[(B(T)-K)^+ 1_{[\Lambda<\lambda(K)]} \mid \Lambda,\mathcal{Z}] - \mathbb{E}[(B(T)-K)1_{[\Lambda<\lambda(K)]} \mid \Lambda,\mathcal{Z}]^+\right] \\ &\le \tfrac{1}{2}\mathbb{E}\left[\mathrm{Var}\left(B(T)1_{[\Lambda<\lambda(K)]} \mid \Lambda,\mathcal{Z}\right)^{1/2}\right] \equiv \varepsilon_1(T,K,\Lambda,\mathcal{Z}) \end{aligned} \tag{2.9}$$

This yields an upper bound which again has to be calculated via a multivariate numerical integration. Nielsen and Sandmann and Vanmaele et al., using only one conditioning variable, go one step further to derive a slightly larger upper bound, that can be calculated in closed-form. Here it is equal to:

$$\begin{aligned} \varepsilon_1(T,K,\Lambda,\mathcal{Z}) &= \tfrac{1}{2}\mathbb{E}[\mathrm{Var}\left(B(T)1_{[\Lambda<\lambda(K)]} \mid \Lambda,\mathcal{Z}\right)^{1/2}] \\ &\le \tfrac{1}{2}\sqrt{\mathbb{E}[\mathrm{Var}\left(B(T) \mid \Lambda,\mathcal{Z}\right)\cdot 1_{[\Lambda<\lambda(K)]}]\cdot \mathbb{E}[1_{[\Lambda<\lambda(K)]}]} \equiv \varepsilon_2(T,K,\Lambda,\mathcal{Z}) \end{aligned} \tag{2.10}$$

Both error estimates yield an upper bound which is equal to $UB_i(T,K,\Lambda,\mathcal{Z}) = LB(T,K,\Lambda,\mathcal{Z}) + \varepsilon_i(T,K,\Lambda,\mathcal{Z})$, for i = 1, 2. It can be calculated in closed-form because we can write:

$$\mathbb{E}[\mathrm{Var}\left(B(T) \mid \Lambda,\mathcal{Z}\right)\cdot 1_{[\Lambda<\lambda(K)]}] = \mathbb{E}\left[\mathbb{E}\left[\mathrm{Var}\left(B(T) \mid \Lambda,\mathcal{Z}\right) \mid \Lambda\right]\cdot 1_{[\Lambda<\lambda(K)]}\right] \tag{2.11}$$

As shown in Nielsen and Sandmann and Vanmaele et al., this expression can be calculated in closed-form. We do not reproduce the formulae here, as it only distracts from the rest of the text and the calculations are exactly the same as in the aforementioned articles. Note that the variance of B(T) given Λ and $\mathcal{Z}$ is zero if the set $\{\Lambda,\mathcal{Z}\}$ contains all random variables within B(T). Then the results above imply that the lower bound exactly coincides with the true value of the Asian basket option. Finally, we mention that Rogers and Shi's upper bound corresponds to the limit of UB_1 for K tending to infinity.

3. The benefits of conditional moment matching

As mentioned in the introduction, many original approximations merely substitute the arithmetic average by a tractable random variable, which has the same first couple of unconditional moments. An example of this is Levy's [1992] approximation, which fits a lognormal random variable to the arithmetic average. These types of approximations typically only work well when volatilities and maturities are low. Furthermore, the size of the error made can not easily be estimated. Here we show that conditional moment matching does yield an analytical error estimate. The proposed approximation follows from:

$$\begin{aligned}\tilde{c}_B(T,K,\Lambda,\mathcal{Z}) &= \mathbb{E}[\left(\tilde{B}(T)-K\right)^+ 1_{[\Lambda<\lambda(K)]} + \left(B(T)-K\right)1_{[\Lambda\geq\lambda(K)]}] \\ &\equiv \tilde{c}_1(T,K,\Lambda) + c_2(T,K,\Lambda)\end{aligned} \tag{3.1}$$

i.e. it again exists of an approximating part and an exact part. For $\Lambda \geq \lambda(K)$ we can take our approximating random variable $\tilde{B}(T)$ to be equal to B(T), yielding the exact c_2-part. For Λ smaller than $\lambda(K)$, we have to make an approximation. Given certain criteria that $\tilde{B}(T)$ must fulfill, which follow in the next theorem, we can find an analytical error estimate as derived in Lord [2005]. Here we extend this result to allow for multiple conditioning variables.

Theorem:
If we impose the following two conditions on the approximating random variable $\tilde{B}(T)$:

$$\begin{aligned}\mathbb{E}[\tilde{B}(T) \mid \Lambda=\lambda, \mathcal{Z}=z] &= \mathbb{E}[B(T) \mid \Lambda=\lambda, \mathcal{Z}=z] \\ \mathrm{Var}[\tilde{B}(T) \mid \Lambda=\lambda, \mathcal{Z}=z] &\leq \mathrm{Var}[B(T) \mid \Lambda=\lambda, \mathcal{Z}=z]\end{aligned} \tag{3.2}$$

for $\lambda \in (-\infty,\lambda(K))$, the resulting approximation in (3.1) lies between $LB(T,K,\Lambda,\mathcal{Z})$ and $UB_i(T,K,\Lambda,\mathcal{Z})$.

Proof:
The proof follows along the same lines as (2.9)-(2.10):

$$\begin{aligned}0 &\leq \tilde{c}_B(T,K,\Lambda) - LB(T,K,\Lambda,\mathcal{Z}) \\ &= \mathbb{E}\left[\mathbb{E}[\left(\tilde{B}(T)-K\right)^+ 1_{[\Lambda<\lambda(K)]} \mid \Lambda,\mathcal{Z}] - \mathbb{E}[\left(\tilde{B}(T)-K\right)1_{[\Lambda<\lambda(K)]} \mid \Lambda,\mathcal{Z}]^+\right] \\ &\leq \tfrac{1}{2}\mathbb{E}\left[\mathrm{Var}\left(\tilde{B}(T)1_{[\Lambda<\lambda(K)]} \mid \Lambda,\mathcal{Z}\right)^{1/2}\right] \leq \varepsilon_1(T,K,\Lambda,\mathcal{Z})\end{aligned} \tag{3.3}$$

It is clear that the first equality holds, due to the construction in (3.1) and the fact that the conditional moments are equal for $\Lambda \leq \lambda(K)$. The rest of the derivation is similar to (2.9)-(2.10). It immediately follows that:

$$LB(T,K,\Lambda,\mathcal{Z}) \leq \tilde{c}_B(T,K,\Lambda,\mathcal{Z}) \leq UB_1(T,K,\Lambda,\mathcal{Z}) \tag{3.4}$$

which concludes the proof of the theorem. □

This theorem directly motivates why it is good to match conditional moments. Intuitively we can indeed expect to obtain better results than by just matching unconditional moments. The above theorem gives a rigorous (and typically sharp) error bound for this. Note that the moments do not have to be exactly matched – the conditional variance may actually be smaller. Approximations satisfying (3.1) and (3.2) are dubbed partially exact and bounded (PEB) approximations. The lower bound LB(T,K,Λ,$\mathcal{Z}$) is a special case hereof. In Vanmaele, Deelstra and Liinev [2004] another route is attempted. Without delving into details, they construct an approximation via a (conditionally) convex combination of the lower bound and the partially exact and comonotonic upper bound (PECUB), the so-called LBPECUB approximation. The conditional weights are chosen by ensuring that the first two conditional moments are matched exactly. As such, it satisfies the criteria for it to be a PEB approximation, and hence it is bounded above by the UB_1 as well of course the PECUB upper bound.

We note that in practice the approximating part in (3.1) will have to be calculated via a numerical integration. From a computational point of view one would therefore not like to use too many conditioning variables. Typically one conditioning variable may already be more than enough, as has been shown in Lord [2005] for a pure Asian option, and as we will demonstrate for a pure basket option in the next and final section.

4. Numerical illustration and conclusions

To illustrate the effectivity of conditional moment matching we will here provide a numerical example for a pure basket option. The example has been taken from Milevsky and Posner [1998], and also features in Vanmaele, Deelstra and Liinev [2004]. The basket underlying the option is the weighted average of the normalized G-7 stock indices. Weights, volatilities, dividend yields and correlations can be found in the tables below.

Country	**Index**	**Weight**	**Volatility**	**Dividend yield**
Canada	TSE 100	10%	11.55%	1.69%
Germany	DAX	15%	14.53%	1.36%
France	CAC 40	15%	20.68%	2.39%
U.K.	FTSE 100	10%	14.62%	3.62%
Italy	MIB 300	5%	17.99%	1.92%
Japan	Nikkei 225	20%	15.59%	0.81%
U.S.	S&P 500	25%	15.68%	1.66%

Table 1: Weights, volatilities and dividend yields of the basket

	Canada	*Germany*	*France*	*U.K.*	*Italy*	*Japan*	*U.S.*
Canada	1	0.35	0.1	0.27	0.04	0.17	0.71
Germany		1	0.39	0.27	0.5	-0.08	0.15
France			1	0.53	0.7	-0.23	0.09
U.K.				1	0.45	-0.22	0.32
Italy					1	-0.29	0.13
Japan						1	-0.03
U.S.							1

Table 2: Upper triangular part of the instantaneous correlation matrix between the various assets

As we use the normalized values of the indices, this effectively means we assume the initial spot value equals 1 for each index. In the following table we compare the lower and upper bounds using one or two conditioning variables to the "true" value obtained from a Monte Carlo simulation with 5.000.000 paths, using antithetic variables and using the geometric basket as the control variate. Results are only shown for the most extreme example in Vanmaele et al., namely for a maturity of 10 years. Vanmaele et al. only considered three strike prices, 0.95, 1 and 1.05. However, the forward price of the basket (the mean of its unconditional distribution) can be calculated as 1.578288, so that we found it important to include higher strike prices in the table as well. The choices for conditioning variables are the same as in this article – the first conditioning variable is FA_1 (cf. (2.3)), the second is similar to FA_2, apart from the fact that the sign of the one but last Brownian motion is reversed.

Strike	**MC**	**(StdErr)**	**LB_{FA1}**	**UB_{FA1}**	**$LB_{FA1,FA2*}$**	**$UB_{FA1,FA2*}$**
0.95	33.6590	(0.0036)	33.6221	33.8765	33.6461	33.7814
1	31.1333	(0.0037)	31.0758	31.3974	31.1103	31.2849
1.05	28.6603	(0.0038)	28.5870	28.9879	28.6335	28.8553
1.25	19.6748	(0.0042)	19.5002	20.3444	19.6046	20.1044
1.5	11.1785	(0.0045)	10.8999	12.5203	11.0481	12.0654
1.578288	9.1918	(0.0044)	8.8975	10.7762	9.0927	10.2872

Table 3: Upper and lower bounds based on one or two conditioning variables

The upper bounds are the UB_2 upper bounds, see equation (2.10). Indeed, conditioning on more random variables sharpens the lower and upper bounds considerably, as was already demonstrated in Vanmaele et al., but is now also apparent from the new upper bound.

To show that conditional moment matching actually works remarkably well, we compare various conditional moment matching approximations to the true value of the option. As mentioned earlier, the LBPECUB approximations considered in Vanmaele, Deelstra and Liinev [2004] are convex combinations of the lower bound and the PECUB upper bound. Results in their paper were shown for using the geometric average as the conditioning variable. Two distinctions can be made on the choice of the weights for each bound: $z(\lambda)$ indicates that the first two conditional moments are matched exactly (yielding a PEB approximation as noted in section 3), whereas z^u indicates that the lower bound and the PECUB upper bound are weighted using a global weight stemming from another approximation in Vyncke, Goovaerts and Dhaene [2003]. The latter is not a conditional moment matching approximation, but it works rather well.

The Curran2M+ and Curran3M+ approximations are PEB approximations considered in Lord [2005] that fit a shifted lognormal random variable to the basket. The 2M+ approximation considered here uses a shift equal to the conditioning variable Λ_{FA1}; the remaining two parameters are chosen such that the first two conditional moments are matched exactly. The 3M+ approximation is a slight take on this: the shift is now also considered as a parameter, so that the first three conditional moments can be fit exactly. In both approximations we condition on Λ_{FA1}.

Strike	**MC**	**(StdErr)**	**$LBPECUB_{GA}$** $z(\lambda)$	z^u	**2M+**	**3M+**
0.95	33.6590	(0.0036)	33.6379	33.6543	33.6682	33.6612
1	31.1333	(0.0037)	31.0971	31.1239	31.1389	31.1318
1.05	28.6603	(0.0038)	28.6149	28.6554	28.6693	28.6631
1.25	19.6748	(0.0042)	-	-	19.6704	19.6769
1.5	11.1785	(0.0045)	-	-	11.1606	11.1768
1.578288	9.1918	(0.0044)	-	-	9.1791	9.1919

Table 4: Several approximations for the value of a basket option

We did not get round to implementing the $LBPECUB_{GA}$ approximation ourselves, so that we here only reproduce the values given in Vanmaele et al. They only considered strike values up to 1.05; as the forward price of the basket is 1.578288 for a 10-year contract, we also found it important to consider slightly higher strike prices. As can be seen from the table, the 3M+ approximation seems to give results that are very close to the true values – and this has only been achieved by using one conditioning variable. The conditional moment matching approximation of Vanmaele et al., using $z(\lambda)$, seems to yield too low values. However, their approximation which uses z^u is a clear contender, yielding results which are comparable to those of the 2M+ approximation. Considering the computational effort, which has been investigated in Lord [2005], we have a slight overall preference for the 2M+ approximation, although this is of course subject to discussion.

Concluding, in this paper we reviewed the conditioning approaches of Rogers and Shi [1995] and Curran [1994]. Rogers and Shi's lower and (sharpened) upper bounds, as well as the PEB approximations of Lord [2005], have been extended along the lines of Vanmaele, Deelstra and Liinev [2004] to allow for multiple conditioning variables. Finally, we have shown that the LBPECUB convex combination of the lower bound and the PECUB upper bound, considered in Vanmaele et al., is indeed a PEB approximation, and as such is bounded from above by the (sharpened) Rogers and Shi upper bound. In a numerical example the effectivity of conditional moment matching has been demonstrated.

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