

# Monte Carlo Pricing in the Schöbel-Zhu Model and its Extensions

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## Abstract

In this paper we propose a simulation algorithm for the Schöbel-Zhu (1999) model and its extension to include stochastic interest rates, the Schöbel-Zhu-Hull-White model as considered in Van Haastrecht et al. (2009). Both schemes are derived by analyzing the lessons learned from the Andersen scheme on how to avoid the so-called leaking correlation phenomenon in the simulation of the Heston (1993) model. All introduced schemes are Exponentially Affine in Expectation (EAE), which greatly facilitates the derivation of a martingale correction. In addition we study the regularity of each scheme. The numerical results indicate that our scheme consistently outperforms the Euler scheme. For a special case of the Schöbel-Zhu model which coincides with the Heston model, our scheme performs similarly to the QE-M scheme of Andersen (2008). The results reaffirm that when simulating stochastic volatility models it is of the utmost importance to match the correlation between the asset price and the stochastic volatility process.

Keywords: Stochastic volatility, Stochastic interest rates, Schöbel-Zhu, Heston, Hull-White, discretisation.

## 1 Introduction

Stochastic volatility models nowadays have become the de facto standard to price and hedge complex financial products; in derivative models the behaviour financial derivatives is usually modeled by stochastic differential equations that jointly describe the movements of the underlying financial assets such as the stock prices, stock variances and interest rates. Though certain models yield closed-form solutions for some products, the vast majority cannot be priced in closed-form. Nonetheless, Monte Carlo methods provide a popular and flexible pricing alternative to value such exotic derivatives. Due to technical advances such as multi-processor programming, increasing computational power and variance reduction techniques, Monte Carlo techniques are expected to become even more widely applicable in the near future. Taking these advances into account, Monte Carlo techniques are still computationally relatively expensive, hence much attention goes out to efficient simulation schemes aiming to minimize the computational efforts whilst retaining a high degree of accuracy.

In the last decennium the literature on efficient simulation schemes for stochastic volatility evolved. Approaches to price derivatives with stochastic volatility models were described in [Hull and White \(1987\)](#), [Stein and Stein \(1991\)](#), [Heston \(1993\)](#) and the [Schöbel and Zhu \(1999\)](#) model. The latter

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two models stand out for allowing the stochastic volatility to be correlated with the underlying asset, whilst still allowing for a closed-form formulas for most vanilla options used in the model's calibration. Discretization schemes for models have been described by several authors, for example [Jäckel \(2002\)](#), [Glasserman \(2003\)](#), [Kahl and Jäckel \(2006\)](#), [Andersen \(2008\)](#), [Lord et al. \(2008\)](#), [Smith \(2008\)](#) and [van Haastrecht and Pelsser \(2009\)](#). Most of these papers focus on efficient discretisation methods for the [Heston \(1993\)](#) model, paying particular attention to the discretisation of the underlying square-root variance process. [Andersen \(2008\)](#) was the first to make the key observation that for any discretisation scheme of the [Heston \(1993\)](#) model it is crucially important to match the correlation between the underlying and the variance process as close as possible. Simple Euler schemes which do not take this into account, suffer from the so-called 'leaking correlation' phenomenon.

In this paper simulation schemes are presented for the [Schöbel and Zhu \(1999\)](#) (SZ) stochastic volatility model and its extensions. Instead of only focussing on the simulation of the volatility process, which in the case of the SZ model is normally distributed and hence can easily be simulated exactly, like Andersen we also pay particular attention to the aforementioned 'leaking correlation' issue. It appears that this issue is a general problem in the simulation of stochastic volatility models. As we aim for our analysis to be as broadly applicable as possible, we also consider an extension of the SZ model which incorporates stochastic interest rates: the Schöbel-Zhu-Hull-White (SZHW) model, as considered in [van Haastrecht et al. \(2009\)](#). This extension combines the SZ model with the 1-factor Gaussian interest rate model of [Hull and White \(1993\)](#), allowing for a general correlation structure between all processes. This is closely related to the recent advances in the development of a market for long-maturity European options in equity and exchange rate derivatives, showing liquid quotes for European options ranging up to 15 years, for which maturities we feel a model including stochastic interest rates is more suitable. Finally, we note that the methods presented here also facilitate the pricing of interest rate derivatives in the context of stochastic volatility Libor Market Models, e.g. see [Zhu \(2007\)](#).

The remainder of the paper is organized as follows. First, the SZ model is described in Section 2. Section 3 analyzes the problem of leaking correlations in the [Schöbel and Zhu \(1999\)](#) and [Heston \(1993\)](#) stochastic volatility models. In Section 4 discretisation schemes are presented for the SZ model. These results are extended with stochastic interest rates in Section 5. In Section 6 numerical examples are worked out, showing the impact of leaking correlations in Monte Carlo methods for stochastic volatility models. Conclusions are given in Section 7.

## 2 The Schöbel-Zhu model

The risk-neutral log-asset price dynamics of the [Schöbel and Zhu \(1999\)](#) model read

$$d \ln x(t) = -\frac{1}{2}v^2(t)dt + v(t)dW_x(t), \quad \ln x(0) = \ln(x_0), \quad (1)$$

$$dv(t) = \kappa(\psi - v(t))dt + \tau dW_v(t), \quad v(0) = v_0, \quad (2)$$

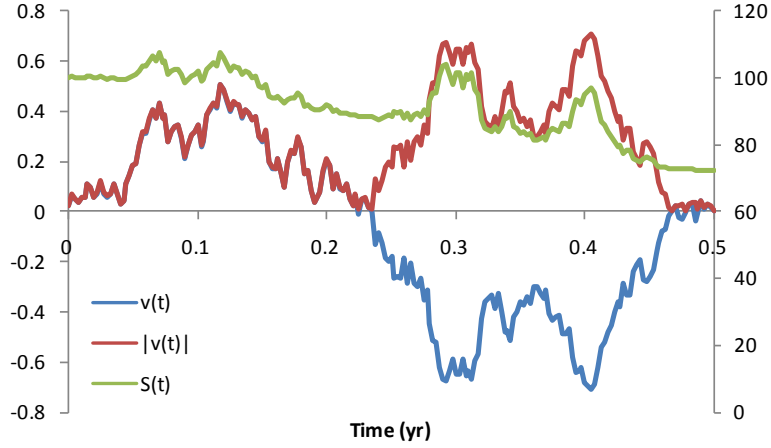
$$dv^2(t) = 2\kappa\left(\frac{\tau^2}{2\kappa} + \psi v(t) - v^2(t)\right)dt + 2\tau v(t)dW_v(t), \quad v^2(0) = v_0^2, \quad (3)$$

where  $\kappa, \psi, \tau$  are positive parameters corresponding to the mean reversion, long-term volatility and volatility of the volatility process and with  $W_x(t)$  and  $W_v(t)$  two Brownian motion under the risk-neutral measure  $Q$  with linear correlation coefficient  $\rho_{xv}$ . At first sight, one curious property of the

model is that the volatility process  $v(t)$  affects the sign of the instantaneous correlation between  $v(t)$  and  $\ln x(t)$ . Indeed, one can show that

$$\text{Corr}(d \ln x(t), dv(t)) = \frac{\rho_{xv} v(t) \tau}{\sqrt{v^2(t) \tau^2}} = \rho_{xv} \text{sgn}(v(t)), \quad (4)$$

This effect is visualized in , where we have plotted a sample path of  $x(t)$ ,  $v(t)$  and  $|v(t)|$ .



**Figure 1:** Sample path of  $x(t)$ ,  $v(t)$  and  $|v(t)|$ . SZ parameters are  $\kappa = \tau = 1$ ,  $v(0) = \psi = 25\%$ ,  $x(0)=100$ .

Indeed, when  $v(t)$  is negative and decreasing, the asset price is increasing, contrary to what one would expect from the parameter configuration. The key lies therein that  $v(t)$  should not be interpreted as the volatility of the underlying asset<sup>5</sup>. It is merely a latent variable which drives the true volatility of the asset, the true volatility being defined as the square root of the variance. If one applies the Ito-Tanaka theorem to derive the dynamics of  $|v(t)|$ , one can indeed show that

$$\text{Corr}(d \ln x(t), d|v(t)|) = \frac{\rho_{xv} |v(t)| \tau}{\sqrt{v^2(t) \tau^2}} = \rho_{xv}, \quad (5)$$

as we would like it to be. In the following section, discretisation schemes for the SZ model, the asset price and the stochastic volatility process.

### 3 Leaking correlation in stochastic volatility models

One of the major problems Andersen (2008) signalled with Euler schemes of the Heston (1993) model, see Lord et al. (2008) for an overview, is their inability to generate a correlation between the increments of the asset and the stochastic volatility processes which resembles that of the true process. As the correlation parameter in stochastic volatility models is an important determinant of the skew in implied volatilities, one can imagine that not being able to match this parameter can lead to a significant mispricing of options with strikes that are further away from the at-the-money level.

<sup>5</sup>One should preferably write 'volatility', but we will refrain from doing this unless it causes confusion

Such problems in the Heston model are partially caused by the fact that an Euler discretisation tries to approximate a Feller process, which is guaranteed to be positive, by a Gaussian process. While stochastic volatility in the [Schöbel and Zhu \(1999\)](#) (SZ) model is itself Gaussian, we will show that 'leaking correlation' as this phenomenon has been dubbed, is still an issue. Before we can design an effective simulation scheme for the SZ model and its extensions, we will have to return to the Heston model and pinpoint exactly why Andersen's simulation schemes are successful in reproducing the right correlation.

In this section we will focus on a special case of the SZ model where the long-term level of mean reversion for the volatility  $v(t)$ ,  $\psi$ , equals zero. This special case also happens to be a special case of the Heston model, which can be seen from the dynamics of  $v^2(t)$

$$dv^2(t) = (\tau^2 - 2\kappa v^2(t))dt + 2\tau v(t)dW_v(t). \quad (6)$$

In this case the Heston and SZ parameters are related as follows

$$\kappa_H \mapsto 2\kappa, \quad \psi_H \mapsto \frac{\tau^2}{2\kappa}, \quad \tau_H \mapsto 2\tau. \quad (7)$$

Recall that from (2) we can easily see that the volatility process follows a standard Gaussian distribution. When  $\psi = 0$ , we can write

$$v(t + \Delta) = K_1 v(t) + K_2 Z_v, \quad (8)$$

with

$$K_1 = e^{-\kappa\Delta}, \quad \text{and } K_2 = \tau \sqrt{\frac{1 - e^{-2\kappa\Delta}}{2\kappa}}. \quad (9)$$

Turning to the log-asset price, integrating the SDE in (1) yields

$$\ln x(t + \Delta) = \ln x(t) - \frac{1}{2} \int_t^{t+\Delta} v^2(u)du + \rho_{xv} \int_t^{t+\Delta} v(u)dW_v(u) + \widehat{\rho}_{xv} \int_t^{t+\Delta} v(u)d\widetilde{W}_x(u), \quad (10)$$

where  $W_v$  and  $\widetilde{W}_x$  are independent Brownian motions and  $\widehat{\rho}_{xv} := \sqrt{1 - \rho_{xv}^2}$ . Using an Euler discretisation, this would become

$$\ln x(t + \Delta) = \ln x(t) - \frac{1}{2}v^2(t)\Delta + v(t)\sqrt{\Delta}\left(\rho_{xv}Z_v + \sqrt{1 - \rho_{xv}^2}Z_x\right), \quad (11)$$

with  $Z_v, Z_x$  standard normally distributed random variables. Conditional upon  $x(t)$  and  $v(t)$ , the correlation between  $\ln x(t + \Delta)$  and  $v(t)$  equals

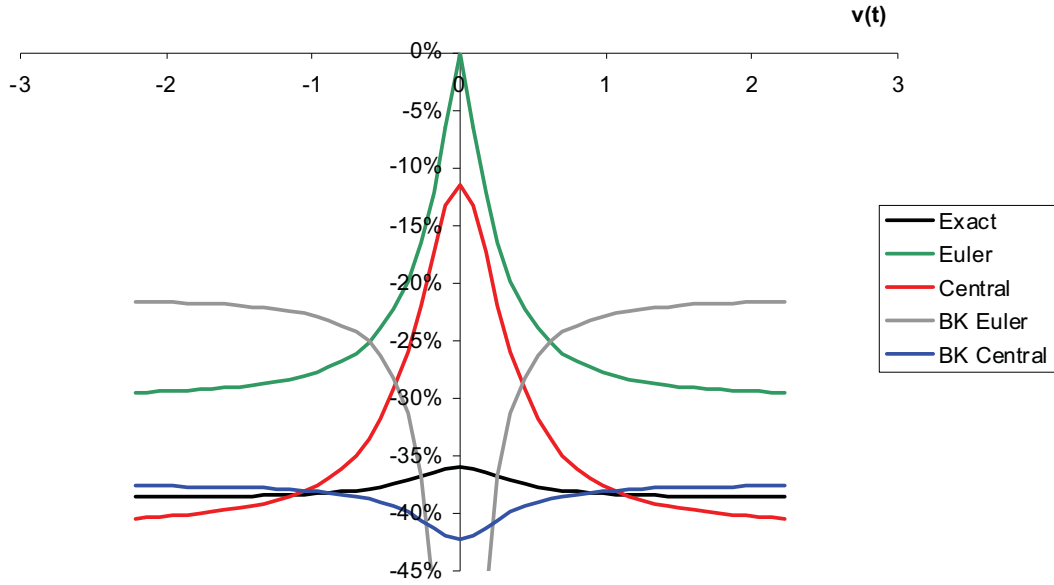
$$\begin{aligned} \text{Corr}_t[\ln x(t + \Delta), v(t + \Delta)] &= \frac{\text{Cov}_t[\ln x(t + \Delta), v(t + \Delta)]}{K_2|v(t)|\sqrt{\Delta}} \\ &= \frac{\text{Cov}_t[\ln x(t + \Delta), Z_v]}{|v(t)|\sqrt{\Delta}} = \rho_{xv} \text{sgn}(v(t)) \end{aligned} \quad (12)$$

So with a naive Euler discretisation it seems there is no 'leaking correlation', as reported in the Heston in [Andersen \(2008\)](#), as this perfectly matches the instantaneous correlation between  $d \ln x(t)$  and  $dv(t)$

in (5). Let us turn to  $v^2(t)$  however, which is a quadratic Gaussian. After some calculations, one can show that in the Euler scheme

$$\text{Corr}_t[\ln x(t + \Delta), v^2(t + \Delta)] = \rho_{xv} \frac{K_1 |v(t)|}{\sqrt{K_1^2 v^2(t) + \frac{1}{2} K_2^2}}, \quad (13)$$

which tends to  $\rho_{xv}$  as  $v(t)$  tends to plus or minus infinity, but can differ substantially when  $v(t)$  is close to zero and even equals zero when  $v(t)$  does. In this sense an Euler discretisation in the SZ model also suffers from leaking correlation in the same way as in the Heston model. This behaviour is visualised in Figure 2.



**Figure 2:** Correlation between  $\ln x(t + \Delta)$  and  $v^2(t + \Delta)$  for various values of the volatility  $v(t)$ . Here we have used the parameters  $\kappa = \tau = 1$ ,  $\rho = -0.3$  and  $\Delta = \frac{1}{4}$ . The central schemes use  $\delta_1 = \delta_2$  for the drift interpolations.

Clearly the correlation from an Euler scheme is far from the exact correlation (see Andersen (2008), appendix A). The question is what the best way is to improve upon the Euler scheme. When we simulate  $x(t + \Delta)$ , we would already have computed  $v(t + \Delta)$ . One possibility is therefore to approximate the integrated variance in (10) using a second order approximation to compute

$$\int_t^{t+\Delta} v^2(u) du \approx [\delta_1 v^2(t) + \delta_2 v^2(t + \Delta)] \Delta, \quad (14)$$

leading to

$$\begin{aligned} \ln x(t + \Delta) = & \ln x(t) - \frac{1}{2} [\delta_1 v^2(t) + \delta_2 v^2(t + \Delta)] \Delta + \rho_{xv} v(t) \cdot (W_v(t + \Delta) - W_v(t)) \\ & + \widehat{\rho}_{xv} \sqrt{\delta_1 v^2(t) + \delta_2 v^2(t + \Delta)} \cdot (\widetilde{W}_x(t + \Delta) - \widetilde{W}_x(t)). \end{aligned} \quad (15)$$

A special case is the central discretisation, where  $\delta_1 = \delta_2 = \frac{1}{2}$ . As can be seen from Figure 2, using a central discretisation does improve the correlation behaviour somewhat, although it is still quite far

from the true correlation. Andersen's discretisation for the log-asset price uses an insight of [Broadie and Kaya \(2006\)](#) which, as can be seen by integrating the SDE of (6) directly, relates the first stochastic integral in (10) in terms of already simulated quantities and the integrated variance

$$\int_t^{t+\Delta} v(u) dW_v(u) = \frac{v^2(t+\Delta) - v^2(t)}{2\tau} - \frac{1}{2}\tau\Delta + \frac{\kappa}{\tau} \int_t^{t+\Delta} v^2(u) du. \quad (16)$$

Substituting this in (10) yields

$$\begin{aligned} \ln x(t+\Delta) &= \ln x(t) + \left( \frac{\rho_{xv}\kappa}{\tau} - \frac{1}{2} \right) \int_t^{t+\Delta} v^2(u) du + \frac{\rho_{xv}}{2\tau} [v^2(t+\Delta) - v^2(t)] \\ &\quad + \sqrt{1 - \rho_{xv}^2} \int_t^{t+\Delta} v(u) d\widetilde{W}_x(u). \end{aligned} \quad (17)$$

Once again we can choose to approximate the integrated variance with an Euler discretisation, or as Andersen does in the Heston model, with a central discretisation. The correlation for both schemes is analysed in Figure 2. As we can see, it is the combination of the insight of [Broadie and Kaya \(2006\)](#) and the central discretisation which bring the correlation much more in line with the true correlation. For this reason all simulation schemes we consider in the remainder of this paper will use Broadie and Kaya's insight, as well as a central discretisation for the integrated variance.

## 4 Simulation in the Schöbel-Zhu model

Having demonstrated in the previous section how to best preserve the correlation structure between the asset and stochastic volatility processes in a special case of the SZ model, we will now formulate our simulation scheme for the full SZ model. In addition, we will demonstrate how to apply a martingale correction such that the forward price of the asset is matched exactly.

For those readers wondering whether an exact simulation of the SZ model is feasible a la [Broadie and Kaya \(2006\)](#), it should be mentioned that, contrary to the Heston model, the increment of the log-asset price process is not normally distributed conditional upon the old and new realizations of the volatility process, and the integrated variance process. In addition to the mentioned realizations, we also need to condition on the integrated volatility process, which complicates matters considerably. Nevertheless, as we have seen in the case of the Heston model, schemes based on a simple drift interpolation method are computationally much more efficient than exact transform-based methods, see [Lord et al. \(2008\)](#), [Andersen \(2008\)](#) or [van Haastrecht and Pelsser \(2009\)](#). From a practical point of view it is therefore not a disadvantage that an exact simulation is not feasible.

### Simulation scheme for the SZ model

As the volatility process  $v$  in (3) follows an Ornstein-Uhlenbeck process, we have the following explicit solution for  $v(t+\Delta)$  (conditional on the time- $t$  filtration):

$$v(t+\Delta) = v(t)e^{-\kappa\Delta} + \int_t^{t+\Delta} \psi e^{-\kappa\Delta} du + \int_t^{t+\Delta} \tau e^{-\kappa\Delta} dW_v(u).$$

As it follows from Ito's isometry that  $(v(t + \Delta)|v(t))$  is normally distributed with mean  $\mu_v := K_1 v(t) + K_2$  and standard deviation  $\sigma_v := K_3$ , a sample of  $v(t + \Delta)|v(t)$  can be obtained by setting

$$v(t + \Delta) = K_1 v(t) + K_2 + K_3 Z_v, \quad (18)$$

where

$$K_1 := e^{-\kappa\Delta} \quad K_2 := \psi(1 - e^{-\kappa\Delta}), \quad (19)$$

$$K_3 := \tau \sqrt{\frac{1}{2\kappa}(1 - e^{-2\kappa\Delta})}, \quad (20)$$

and  $Z_v$  is sample from the standard normal distribution. This can be generated directly and efficiently by 'inverting' the standard normal distribution, e.g. see [Acklam \(2003\)](#). Note that the above sampling of the volatility process, immediately also gives us realizations for the variance process.

As the previous section demonstrated, it is beneficial to apply the [Broadie and Kaya \(2006\)](#) insight and replace  $\int_t^{t+\Delta} v(u) d\widetilde{W}_x^T(u)$  in (10) by expressing it in other model quantities. This can be achieved by integrating (3), leading to

$$\int_t^{t+\Delta} v(u) d\widetilde{W}_v^T(u) = \frac{1}{2\tau} (v^2(t + \Delta) - v^2(t) - \tau^2\Delta + 2\kappa \int_t^{t+\Delta} v^2(u) du - 2\kappa \int_t^{t+\Delta} \psi v(u) du). \quad (21)$$

Substituting (21) in (10) yields

$$\begin{aligned} \ln x(t + \Delta) &= \ln x(t) - \frac{1}{2} \int_t^{t+\Delta} v(u) du + \frac{\rho_{xv} \kappa}{\tau} \int_t^{t+\Delta} (v^2(u) - \psi v(u)) du \\ &\quad + \frac{\rho_{xv}}{2\tau} (v^2(t + \Delta) - v^2(t) - \tau^2\Delta) + \sqrt{1 - \rho_{xv}^2} \int_t^{t+\Delta} v(u) \widetilde{W}_x(u) \end{aligned} \quad (22)$$

As in the previous section, we replace the integrals over the variance and volatility by linear combinations of their realizations at  $t$  and  $t + \Delta$

$$\int_t^{t+\Delta} v^p(u) du | v(t), v(t + \Delta) \approx \delta_1 v^p(t) + \delta_2 v^p(t + \Delta), \quad (23)$$

for  $p \in \{1, 2\}$  and some constants  $\delta_1, \delta_2$ . These constants can be set in several ways: an Euler-like setting would read  $\delta_1 = 1, \delta_2 = 0$ , while a central/mid-point/predictor-corrector method uses  $\delta_1 = \delta_2 = \frac{1}{2}$ . By applying the above drift interpolation method in (22), one obtains the following discretisation scheme

$$\begin{aligned} \ln x(t + \Delta) &= \ln x(t) + C_0 + C_1 v(t) + C_2 v(t + \Delta) + C_3 v^2(t) + C_4 v^2(t + \Delta) \\ &\quad + \sqrt{\delta_1 v^2(t) + \delta_2 v^2(t + \Delta)} C_5 Z_x \end{aligned} \quad (24)$$

with

$$\begin{aligned} C_0 &= -\frac{1}{2}\rho_{xv}\tau\Delta, & C_1 &= -\delta_1\rho_{xv}\frac{\psi\kappa\Delta}{\tau}, & C_2 &= -\delta_2\rho_{xv}\frac{\psi\kappa\Delta}{\tau}, \\ C_3 &= -\frac{1}{2}\delta_1\Delta + \frac{\rho_{xv}}{\tau}\left(\delta_1\kappa\Delta - \frac{1}{2}\right), & C_4 &= -\frac{1}{2}\delta_2\Delta + \frac{\rho_{xv}}{\tau}\left(\delta_2\kappa\Delta + \frac{1}{2}\right), & C_5 &= \sqrt{1 - \rho_{xv}^2}\sqrt{\Delta}. \end{aligned}$$

Despite the fact the scheme is based on the exact solution of the asset and volatility processes, the discretisation for the log-asset is in general not a martingale, and its net drift away from a martingale can be significant for certain parameter choices. In the following section we show how to enforce this martingale condition. As (24) is exponentially affine after we exponentiate and take expectations with respect to the Gaussian random variates, we will refer to this scheme as an Exponentially Affine in Expectation (EAE) scheme. This property will prove to be very convenient in enforcing the exact martingale condition.

#### 4.1 Martingale correction, regularity

As discussed in Andersen and Piterbarg (2007), the continuous-time asset price process  $x(t)$  might not have finite higher moments, but the stock price will always be a martingale under its natural measure,

$$\mathbb{E}^{Q^T} [x(t + \Delta) | \mathcal{F}_t] = x(t) < \infty. \quad (25)$$

If we replace  $x(t + \Delta)$  by its discretisation, the martingale condition is no longer satisfied. Though the net drift away from the martingale is controllable by reducing the size of the time step, its size, as mentioned, can be significant depending on the parameters of the model. Following Glasserman and Zhoa (2000) and Andersen (2008), we investigate whether it is possible to exactly satisfy this martingale property. Additionally, we look at the regularity of the discretisation scheme: that is, we look whether there might parameter values where the  $\tilde{x}$ -process might blow up, see e.g. Andersen and Piterbarg (2007) for a general discussion of this phenomenon in stochastic volatility models.

First of all notice that by the tower law of conditional expectations, we have

$$\mathbb{E}^{Q^T} [\tilde{x}(t + \Delta) | \mathcal{F}_t] = \mathbb{E} \left\{ \mathbb{E}^{Q^T} [\tilde{x}(t + \Delta) | \mathcal{F}_t, v(T)] | \mathcal{F}_t \right\}, \quad (26)$$

hence for the martingale condition (25) to hold, we need the latter expectation to equal  $\tilde{x}(t)$ ; using the moment-generating function of the normal distribution, we have the following for the discretized stock price  $\tilde{x}(t + \Delta)$ ,

$$\begin{aligned} \tilde{x}(t + \Delta) &= \tilde{x}(t) \exp[C_0^* + C_1 v(t) + C_3 v^2(t)] \mathbb{E} \left\{ \exp[C_2 v(t + \Delta) + C_4 v^2(t + \Delta)] \right. \\ &\quad \left. \mathbb{E}^{Q^T} (\exp[\sqrt{\delta_1 v^2(t) + \delta_2 v^2(t + \Delta)} C_5 Z_x] | \mathcal{F}_t, v(t + \Delta)) \right\} \\ &= \tilde{x}(t) \exp[C_0^* + C_1 v(t) + C_3 v^2(t)] \\ &\quad \mathbb{E} \left[ \exp[C_2 v(t + \Delta) + C_4 v^2(t + \Delta) + \frac{1}{2} C_5^2 (\delta_1 v^2(t) + \delta_2 v^2(t + \Delta))] \right]. \end{aligned} \quad (27)$$

As mentioned earlier, this is where the EAE property of the scheme becomes apparent. We are left with evaluating the expectation of an exponentially affine form. Taking the  $\mathcal{F}_t$  measurable terms out

of the expectation, and dividing by  $\tilde{x}(t)$ , we thus find that the following expectation needs to be satisfied for the martingale condition,

$$\begin{aligned} 1 &= \exp\left[C_0^* + D_1v(t) + D_3v^2(t)\right] \mathbb{E}^{Q^T} \left\{ \exp\left[D_2v(t + \Delta) + D_4v^2(t + \Delta)\right] \middle| \mathcal{F}_t \right\} \\ &= \exp\left[C_0^* + D_1v(t) + D_3v^2(t)\right] \Psi_H(1) \end{aligned} \quad (28)$$

where  $\Psi_H(t)$  denotes the moment-generating function of the (discretized) process

$$H := D_2v(t + \Delta) + D_4v^2(t + \Delta), \quad (29)$$

evaluated in the point  $t$ , with

$$D_1 := C_1, \quad D_2 := C_2, \quad D_3 := C_3 + \frac{1}{2}(1 - \rho_{xv}^2)\delta_1\Delta, \quad D_4 := C_4 + \frac{1}{2}(1 - \rho_{xv}^2)\delta_2\Delta. \quad (30)$$

If the regularity condition  $\Psi_H(1) < \infty$  is satisfied, the martingale condition (i.e. equation (28)) can be satisfied by setting

$$C_0^* := -D_1v(t) - D_3v^2(t) - \ln\left(\Psi_H(1)\right). \quad (31)$$

It now remains to determine the moment-generating function of the random variable  $H$  and investigate its existence. To this end, we will use the following corollary.

**Corollary 4.1** *Let  $X$  be a normally distributed random variable with mean  $\mu$  and variance  $\sigma^2$ , furthermore let  $p$  and  $q$  be two positive constants. Then provided that the regularity condition  $uq\sigma^2 < 1$  is satisfied, the moment-generating function of  $Y := pX + \frac{q}{2}X^2$  is given by*

$$\mathbb{E} \exp(uY) = \exp\left(-\frac{p^2}{2q}\right) \frac{\exp\left(\frac{\lambda u \frac{q\sigma^2}{2}}{1 - uq\sigma^2}\right)}{\sqrt{1 - uq\sigma^2}}, \quad (32)$$

with:

$$\lambda = \left(\frac{\mu + \frac{p}{q}}{\sigma}\right)^2. \quad (33)$$

**Proof** For example see [Johnson et al. \(1994\)](#).

Since the volatility process, conditional upon  $\mathcal{F}_t$ , is normally distributed, we can immediately use Corollary 4.1 with  $p = D_2$  and  $q = 2D_4$ . Provided that  $2D_4\sigma_v^2 < 1$ , we find that  $\Psi_H(1)$  is given by

$$\Psi_H(1) = \exp\left(-\frac{D_2^2}{4D_4}\right) \frac{\exp\left(\frac{\lambda_v D_4 K_3^2}{1 - 2D_4 K_3^2}\right)}{\sqrt{1 - 2D_4 K_3^2}}. \quad (34)$$

with

$$\lambda_v := \left(\frac{\mu_v + \frac{D_2}{2D_4}}{K_3}\right)^2 = \left(\frac{v(t)K_1 + K_2 + \frac{D_2}{2D_4}}{K_3}\right)^2, \quad (35)$$

with  $K_1, K_2$  as defined in (19) and (20).

The following proposition applies the above result to the martingale correction in (24) and the corresponding regularity condition.

**Proposition 4.2** *The regularity of the simulation scheme (24) holds if and only if the following regularity condition is satisfied.*

$$\frac{\tau^2}{\kappa} \left(1 - e^{-2\kappa\Delta}\right) \left[ -\frac{1}{2} \rho_{xv}^2 \delta_2 \Delta + \frac{\rho_{xv}}{\tau} \left( \delta_2 \kappa \Delta + \frac{1}{2} \right) \right] < 1, \quad (36)$$

Given that this condition is satisfied, we can ensure the martingale property in the SZ-scheme of (24) by replacing the constant  $C_0$  by

$$C_0^* = E_0 + E_1 v(t) + E_2 v^2(t), \quad (37)$$

with

$$\begin{aligned} E_0 &:= \frac{1}{2} \ln \left( 1 - 2D_4 K_3^2 \right) - \frac{D_4 \left( K_2 + \frac{D_2}{2D_4} \right)^2}{1 - 2D_4 K_3^2}, & E_1 &:= -D_1 - \frac{2D_4 K_1 \left( K_2 + \frac{D_2}{2D_4} \right)}{1 - 2D_4 K_3^2}, \\ E_2 &:= -D_3 - \frac{D_4 K_1^2}{1 - 2D_4 K_3^2} \end{aligned} \quad (38)$$

and where  $K_1, K_2, K_3$  as defined in (19), (20) and  $D_1, \dots, D_4$  in (30).

**Proof** Follows immediately from the results above.

**Remark** We note that (36) is not restrictive; for negative  $\rho_{xv}$  (which is more than often the case in option markets), the condition is automatically satisfied. However for (strictly) positive  $\rho_{xv}$  the condition (36) imposes a limit on the size of the time step. Nonetheless, for practical sizes of the time step (e.g.  $\Delta = \frac{1}{4}$ ), it is unlikely that the regularity condition (36) will be violated.

## 5 Monte Carlo pricing under Stochastic Interest Rates

The Schöbel-Zhu-Hull-White (SZHW) model as introduced in [van Haastrecht et al. \(2009\)](#) extends the [Schöbel and Zhu \(1999\)](#) model for stochastic volatility with [Hull and White \(1993\)](#) stochastic interest rates. It can for example be used to value equity options or equity-interest rate hybrids. It models three key variables which are allowed to be correlated with each other: the stock price  $x(t)$ , the [Hull and White \(1993\)](#) interest rate process  $r(t)$  and the stochastic volatility of the stock which follows an Ornstein-Uhlenbeck process conform [Schöbel and Zhu \(1999\)](#). The risk-neutral asset price dynamics of the SZHW model hence read:

$$dx(t) = x(t)r(t)dt + x(t)v(t)dW_x(t), \quad x(0) = x_0, \quad (39)$$

$$dr(t) = (\theta(t) - ar(t))dt + \sigma dW_r(t), \quad r(0) = r_0, \quad (40)$$

$$dv(t) = \kappa(\psi - v(t))dt + \tau dW_v(t), \quad v(0) = v_0, \quad (41)$$

where  $a, \sigma, \kappa, v_0, \psi, \tau$  are positive parameters which can be inferred from market data and correspond to the mean reversion and volatility of the short rate process, and the mean reversion, long-term volatility and volatility of the volatility process respectively. The quantity  $r_0$  and the deterministic function  $\theta(t)$  are used to match the currently observed term structure of interest rates, e.g. see [Hull and White \(1993\)](#). Finally,  $\tilde{W}(t) = (W_x(t), W_r(t), W_v(t))$  denotes a Brownian motion under the risk-neutral measure  $\mathbb{Q}$  with covariance matrix:

$$\text{Var}(\tilde{W}(t)) = \begin{pmatrix} 1 & \rho_{xr} & \rho_{xv} \\ \rho_{xr} & 1 & \rho_{rv} \\ \rho_{xv} & \rho_{rv} & 1 \end{pmatrix} t \quad (42)$$

Though one can price vanilla options in SZHW model by transforming the characteristic function of the log-asset price (e.g. see [van Haastrecht et al. \(2009\)](#)), sometimes there is a need to price more complex securities, such as path-dependent or multi-asset securities, for which one has to resort to e.g. Monte Carlo simulation. In this section we will present a simulation scheme for the SZHW model, based on the insights of the previous section.

First of all, instead of looking at these dynamics under the risk-neutral bank account measure we change the underlying probability measure to evaluate this expectation under the  $T$ -forward probability measure  $Q^T$  (e.g. see [Geman et al. \(1996\)](#)). Effectively this reduces the dimension of the Monte Carlo simulation as one can eliminate the path dependency of the stochastic interest rates in discounting future cash flows as stochastic discounting is done using  $T$ -forward bond instead of the money market account. For the analysis of the SZHW simulation it makes no real difference whether we perform the analysis under the risk-neutral or  $T$ -forward measure, as the simulation issues and corresponding solutions are the same under both setups. As we would like to focus on the SZHW specific issues, we therefore prefer with working under the  $T$ -forward measure. For an exact joint simulation of the short rate and the stochastic discount factor in the [Hull and White \(1993\)](#) model, which follows a bivariate normal distribution, see e.g. [Glasserman \(2003\)](#). To this end we define  $y(t, T)$ , the logarithm of the forward stock price  $F(t, T)$ , as

$$y(t, T) := \ln\left(\frac{S(t)}{P(t, T)}\right) = \ln F(t, T). \quad (43)$$

An application of Ito's lemma yields the following asset price dynamics

$$dy(t, T) = -\frac{1}{2}v_F^2(t)dt + v(t)dW_x^T(t) + \sigma B_r(t, T)dW_r^T(t), \quad (44)$$

$$v_F^2(t) := v^2(t) + 2\rho_{xr}v(t)\sigma B_r(t, T) + \sigma^2 B_r^2(t, T), \quad (45)$$

with  $B_r(u, T) := \frac{1}{a}[1 - e^{-a(T-u)}]$  and where the volatility and variance dynamics read

$$dv(t) = \kappa\left(\psi - \frac{\rho_{rv}\sigma\tau}{\kappa}B_r(t, T) - v(t)\right)dt + \tau dW_v^T(t), \quad (46)$$

$$dv^2(t) = 2\kappa\left(\frac{\tau^2}{2\kappa} + \left[\psi - \frac{\rho_{rv}\sigma\tau}{\kappa}B_r(t, T)\right]v(t) - v^2(t)\right)dt + 2\tau v(t)dW_v^T(t). \quad (47)$$

Before turning to the simulation of the asset price dynamics, we first consider the simulation of the Gaussian rate and volatility process which is common in both of the schemes we will consider.

## 5.1 Interest rate and variance simulation

For the Ornstein-Uhlenbeck stochastic volatility process, one has the following solution under the  $T$ -forward measure  $Q^T$ :

$$v(t + \Delta) = v(t)e^{-\kappa\Delta} + \int_t^{t+\Delta} \kappa\xi(u)e^{-\kappa(t+\Delta-u)}du + \int_t^{t+\Delta} \tau e^{-\kappa(t+\Delta-u)}dW_v^T(u) \quad (48)$$

From Ito's isometry we therefore have that  $(v(t + \Delta)|v(t))$  is normally distributed with mean  $\mu_v = K_1v(t) + K_2$  and variance  $\sigma_v^2 = K_3^2$ . A sample of  $v(t + \Delta)|v(t)$  can be obtained by:

$$v(t + \Delta) = K_1v(t) + K_2 + K_3Z_v, \quad (49)$$

with  $Z_\nu$  a standard normal distributed random variable and where:

$$K_1 := e^{-\kappa\Delta}, \quad K_2 := \left(\psi - \frac{\rho_{rv}\sigma\tau}{a\kappa}\right)(1 - e^{-\kappa\Delta}) - \frac{\rho_{rv}\sigma\tau}{a(\kappa + a)}(e^{-a(T-t)-\kappa\Delta} - e^{-a(T-t-\Delta)}), \quad (50)$$

$$K_3 := \tau \sqrt{\frac{1}{2\kappa}(1 - e^{-2\kappa\Delta})}. \quad (51)$$

Though the volatility and the directly related variance process can be simulated from their exact distributions, we need to resort to discretisation methods for a (joint) asset price sampling. We will deal with this in the following sections.

## 5.2 Asset price sampling scheme

Recall that we have the following solution for the SZHW log-asset price solution under the  $T$ -forward measure  $\mathcal{Q}^T$ .

$$y(t + \Delta, T) = y(t, T) - \frac{1}{2} \int_t^{t+\Delta} v_F^2(u) du + \sigma \int_t^{t+\Delta} B_r(u, t) dW_r^T(u) + \int_t^{t+\Delta} v(u) dW_x^T(u) \quad (52)$$

with

$$v_F^2(u) = v^2(u) + 2\rho_{xr}v(u)\sigma B_r(u, T) + \sigma^2 B_r^2(u, T), \quad (53)$$

and where  $W_\nu(u)$ ,  $w_x^T(u)$  and  $w_r^T(u)$  are three correlated Brownian motions. In a Monte Carlo simulation it is often convenient to express these correlated Brownian motions in terms of three orthogonal components  $\widetilde{W}_x^T$ ,  $\widetilde{W}_v^T$  and  $\widetilde{W}_r^T$ , e.g. by using a Cholesky decomposition; the asset dynamics of (52) hence become

$$\begin{aligned} y(t + \Delta, T) &= y(t, T) - \frac{1}{2} \int_t^{t+\Delta} v_F^2(u) du \\ &+ \int_t^{t+\Delta} v(u) d\left(\rho_{xv}\widetilde{W}_v^T(u) + \sqrt{1 - \rho_{xv}^2}\widetilde{W}_x^T(u)\right) \\ &+ \sigma \int_t^{t+\Delta} B_r(u, T) d\left(\rho_{rv}\widetilde{W}_v^T(u) + \omega_{xr}\widetilde{W}_x^T(u) + \sqrt{1 - \rho_{rv}^2 - \omega_{xr}^2}\widetilde{W}_r^T(u)\right), \end{aligned} \quad (54)$$

with

$$\omega_{xr} = \frac{\rho_{xr} - \rho_{xv}\rho_{rv}}{\sqrt{1 - \rho_{xv}^2}}. \quad (55)$$

As Section 4 demonstrated, it is beneficial to apply the [Broadie and Kaya \(2006\)](#) insight and replace  $\int_t^{t+\Delta} v(u) d\widetilde{W}_x^T(u)$  in (54) by expressing it in other model quantities. This can be achieved by integrating (47), leading to

$$\int_t^{t+\Delta} v(u) d\widetilde{W}_v^T(u) = \frac{1}{2\tau} (v^2(t + \Delta) - v^2(t) - \tau^2\Delta + 2\kappa \int_t^{t+\Delta} v^2(u) du - 2\kappa \int_t^{t+\Delta} \xi(u)v(u) du). \quad (56)$$

Substituting (56) in (54) yields

$$\begin{aligned}
y(t + \Delta, T) &= y(t, T) - \frac{1}{2} \int_t^{t+\Delta} v_F^2(u) du + \frac{\rho_{xy}K}{\tau} \int_t^{t+\Delta} (v^2(u) - \xi(u)v(u)) du \\
&\quad + \frac{\rho_{xy}}{2\tau} (v^2(t + \Delta) - v^2(t) - \tau^2 \Delta) + \int_t^{t+\Delta} \left( \sqrt{1 - \rho_{xy}^2} v(u) + \omega_{xr} \sigma B_r(u, T) \right) \widetilde{W}_x^T(u) \\
&\quad + \rho_{rv} \int_t^{t+\Delta} \sigma B_r(u, T) d\widetilde{W}_v^T(u) + \sqrt{1 - \rho_{rv}^2 - \omega_{xr}^2} \int_t^{t+\Delta} \sigma B_r(u, T) d\widetilde{W}_r^T(u).
\end{aligned} \tag{57}$$

This leaves us with three stochastic integrals, which we tackle in order of complexity. We start with last, which follows directly from Ito's isometry

$$\int_t^{t+\Delta} \sigma B_r(u, T) d\widetilde{W}_r^T(u) \sim \sqrt{\int_t^{t+\Delta} \sigma^2 B_r^2(u, T) du} \cdot Z_r, \tag{58}$$

with  $Z_r$  an independent standard normal distributed random variable.

The first integral in (57) follows similarly as:

$$\begin{aligned}
&\int_t^{t+\Delta} \left( \sqrt{1 - \rho_{xy}^2} v(u) + \omega_{xr} \sigma B_r(u, T) \right) \widetilde{W}_x^T(u) \\
&\sim \sqrt{\int_t^{t+\Delta} \left( (1 - \rho_{xy}^2) v^2(u) + 2 \sqrt{1 - \rho_{xy}^2} \omega_{xr} \sigma B_r(u, T) v(u) + \omega_{xr}^2 \sigma^2 B_r^2(u, T) \right) du} \cdot Z_x, \tag{59}
\end{aligned}$$

with  $Z_x$  an independent standard normal distributed random variable. Finally the second integral follows from the fact that the pair  $\left( \int_t^{t+\Delta} \sigma B_r(u, T) d\widetilde{W}_v^T(u), \int_t^{t+\Delta} d\widetilde{W}_v^T(u) \right)$  follows a bivariate normal distribution with correlation  $\rho_{v2}(t, t + \Delta)$  and a conditioning argument

$$\begin{aligned}
\int_t^{t+\Delta} \sigma B_r(u, T) d\widetilde{W}_v^T(u) \Big| \int_t^{t+\Delta} d\widetilde{W}_v^T(u) &\sim \sqrt{G(t, t + \Delta)} \left( \rho_{v2}(t, t + \Delta) Z_v + \sqrt{1 - \rho_{v2}^2(t, t + \Delta)} Z_{v\Delta} \right) \\
\rho_{v2}(t, t + \Delta) &:= \frac{\int_t^{t+\Delta} \sigma B_r(u, T) du}{\sqrt{\Delta \cdot G(t, t + \Delta)}}, \tag{61}
\end{aligned}$$

with  $Z_{v2}$  an independent standard normal random variable.

Having eliminated all stochastic integrals, we are left with deterministic integrals over  $\sigma B_r(u, T)$ ,  $v(u)$  and powers thereof; for the deterministic integrals over  $\sigma B_r(u, T)$  we use the following explicit

solutions:

$$\int_t^{t+\Delta} \sigma B_r(u, T) du = \frac{\sigma}{a} \left[ \Delta - \frac{1}{a} e^{-a(T-t-\Delta)} + \frac{1}{a} e^{-a(T-t)} \right] =: H(t, t + \Delta), \quad (62)$$

$$\begin{aligned} \int_t^{t+\Delta} \sigma^2 B_r^2(u, T) du &= \frac{\sigma^2}{a^2} \left[ \Delta + \frac{1}{2a} e^{-2a(T-t-\Delta)} - \frac{2}{a} e^{-a(T-t-\Delta)} - \frac{1}{2a} e^{-2a(T-t)} + \frac{2}{a} e^{-a(T-t)} \right], \\ &=: G(t, t + \Delta), \end{aligned} \quad (63)$$

whereas we will approximate all integrals over  $v(u)$  by using the predictor-corrector method:

$$\int_t^{t+\Delta} v^p(u) du, \approx (\delta_1 v^p(t) + \delta_2 v^p(t + \Delta)) \Delta, \quad (64)$$

for  $p \in \{1, 2\}$  and some constants  $\delta_1, \delta_2$ .

Collecting all terms once again yields an Exponentially Affine in Expectation (EAE) scheme for the SZHW model

$$y(t + \Delta, T) = y(t, T) + C_0 + C_1 v(t) + C_2 v(t + \Delta) + C_3 v^2(t) + C_4 v^2(t + \Delta) + C_5 Z_x + C_6 Z_v + C_7 Z_{v^2} + C_8 Z_r \quad (65)$$

where:

$$\begin{aligned} C_0 &= -\frac{1}{2} \left[ G(t, t + \Delta) + \rho_{xv} \tau \Delta \right], \\ C_1 &= -\delta_1 \left( \rho_{xr} H(t, t + \Delta) + \rho_{xv} \left[ \frac{\psi \kappa \Delta}{\tau} - \rho_{rv} H(t, t + \Delta) \right] \right), \\ C_2 &= -\delta_2 \left( \rho_{xr} H(t, t + \Delta) + \rho_{xv} \left[ \frac{\psi \kappa \Delta}{\tau} - \rho_{rv} H(t, t + \Delta) \right] \right), \\ C_3 &= -\frac{1}{2} \delta_1 \Delta + \frac{\rho_{xv}}{\tau} \left( \delta_1 \kappa \Delta - \frac{1}{2} \right), & C_4 &= -\frac{1}{2} \delta_2 \Delta + \frac{\rho_{xv}}{\tau} \left( \delta_2 \kappa \Delta + \frac{1}{2} \right), \\ C_5 &= \sqrt{C_{50} + C_{51} v(t) + C_{52} v(t + \Delta) + C_{53} v^2(t) + C_{54} v^2(t + \Delta)}. \\ C_{50} &= \omega_{xr}^2 G(t, t + \Delta), & C_{51} &= 2\delta_1 \omega_{xr} \sqrt{1 - \rho_{xv}^2} H(t, t + \Delta), \\ C_{52} &= 2\delta_2 \omega_{xr} \sqrt{1 - \rho_{xv}^2} H(t, t + \Delta), & C_{53} &= \delta_1 \Delta (1 - \rho_{xv}^2), \\ C_{54} &= \delta_2 \Delta (1 - \rho_{xv}^2), & C_6 &= \rho_{rv} \sqrt{G(t, t + \Delta)} \rho_{vv^2}(t, t + \Delta), \\ C_7 &= \rho_{rv} \sqrt{G(t, t + \Delta)} \sqrt{1 - \rho_{vv^2}^2(t, t + \Delta)}, & C_8 &= \sqrt{1 - \rho_{rv}^2 - \omega_{xr}^2} \sqrt{G(t, t + \Delta)}. \end{aligned}$$

Similar to the SZ scheme (24), the above simulation scheme might have a net drift away from the martingale and violate the (no-arbitrage) martingale property. Nonetheless, in the following section, we show that one can easily enforce this martingale condition by replacing the constant  $C_0$  with  $C_0^*$  of equation (73).

### 5.3 Martingale correction, regularity

In this section, using similar techniques as in Section 4.1, we will investigate how to enforce the martingale property of the discretized asset price process  $\tilde{x}$  in predictor-corrector scheme (65);

furthermore, we investigate the regularity of the proposed discretisation scheme, i.e. we look whether there are parameter values where the  $\tilde{x}$ -process blows up.

By the tower law of conditional expectations, we have the following

$$\mathbb{E}^{Q^T} [\tilde{F}(t + \Delta) | \mathcal{F}_t] = \mathbb{E}^{Q^T} \left\{ \mathbb{E}^{Q^T} [\tilde{F}(t + \Delta) | \mathcal{F}_t, \nu(T)] | \mathcal{F}_t \right\}, \quad (66)$$

hence for the martingale condition (25) to hold, we need  $\tilde{F}(t)$  to equal the latter expectation. We express the inner expectation completely in terms of  $\nu(t)$  and  $\nu(t + \Delta)$  by exponentiating (65), taking the expectation over the independent normal distributions  $Z_x, Z_{v2}$  and  $Z_r$ , and noting from (50) that  $Z_\nu := \frac{\nu(t+\Delta) - \nu(t)K_1 - K_2}{K_3}$ . We obtain the following expression

$$\mathbb{E}^{Q^T} [\tilde{F}(t + \Delta) | \mathcal{F}_t, \nu(T)] = \tilde{F}(t) \exp \left[ D_0 + D_1 \nu(t) + D_2 \nu(t + \Delta) + D_3 \nu^2(t) + D_4 \nu^2(t + \Delta) \right] \quad (67)$$

where

$$\begin{aligned} D_0 &:= C_0^* + \frac{1}{2}C_8^2 + \frac{1}{2}C_7^2 + \frac{1}{2}C_{50} - C_6 \frac{K_2}{K_3}, & D_1 &:= C_1 + \frac{1}{2}C_{51} - \frac{K_1}{K_3}C_6, \\ D_2 &:= C_2 + \frac{1}{2}C_{52} + \frac{1}{K_3}C_6, & D_3 &:= C_3 + \frac{1}{2}C_{53}, & D_4 &:= C_4 + \frac{1}{2}C_{54}. \end{aligned} \quad (68)$$

Once again, due to the EAE property of our scheme, this term is exponentially affine. By substituting (67) in (66), we find that the following condition has to be satisfied for the martingale condition to hold:

$$1 = \mathbb{E}^{Q^T} \left\{ \exp \left[ D_0 + D_1 \nu(t) + D_2 \nu(t + \Delta) + D_3 \nu^2(t) + D_4 \nu^2(t + \Delta) \right] | \mathcal{F}_t \right\}. \quad (69)$$

Taking the  $\mathcal{F}_t$  measurable terms out of the expectation and collecting terms, we obtain

$$\begin{aligned} 1 &= \exp \left[ D_0 + D_1 \nu(t) + D_3 \nu^2(t) \right] \mathbb{E}^{Q^T} \left\{ \exp \left[ D_2 \nu(t + \Delta) + D_4 \nu^2(t + \Delta) \right] | \mathcal{F}_t \right\} \\ &= \exp \left[ D_0 + D_1 \nu(t) + D_3 \nu^2(t) \right] \Psi_H(1) \end{aligned} \quad (70)$$

where  $\Psi_H(t)$  denotes the moment-generating function of the (discretized) process

$$H := D_2 \nu(t + \Delta) + D_4 \nu^2(t + \Delta), \quad (71)$$

evaluated in the point  $t$ . Hence expanding  $D_0$ , we have that for the martingale condition to hold we need

$$1 = \exp \left[ C_0^* + \frac{1}{2}C_8^2 + \frac{1}{2}C_7^2 + \frac{1}{2}C_{50} - C_6 \frac{K_2}{K_3} + D_1 \nu(t) + D_3 \nu^2(t) \right] \Psi_H(1) \quad (72)$$

which (assuming the regularity condition  $\Psi_H(1) < \infty$  is satisfied) can be established by setting

$$C_0^* := -\frac{1}{2}C_8^2 - \frac{1}{2}C_7^2 - \frac{1}{2}C_{50} + C_6 \frac{K_2}{K_3} - D_1 \nu(t) - D_3 \nu^2(t) - \ln \left( \Psi_H(1) \right). \quad (73)$$

As  $\nu(t + \Delta)$  is still Gaussian under  $Q^T$ ,  $\Psi_H(1)$  and its regularity can be determined in a similar fashion to Section 4.1. The following proposition applies the above result to the martingale correction and the regularity of the simulation scheme (65).

**Proposition 5.1** *The regularity of the simulation scheme (65) holds if and only if the regularity condition (36) is satisfied. Given that this condition is satisfied, we can ensure the martingale property in the SZHW-scheme of (65) by replacing the constant  $C_0$  by*

$$C_0^* = E_0 + E_1 v(t) + E_2 v^2(t), \quad (74)$$

where:

$$E_0 := \frac{1}{2} \ln(1 - 2D_4 K_3^2) - \frac{D_4 \left(K_2 + \frac{D_2}{2D_4}\right)^2}{1 - 2D_4 K_3^2} - \frac{1}{2} C_{50} + C_6 \frac{K_2}{K_3} - \frac{1}{2} C_7^8 - \frac{1}{2} C_8^2 + \frac{D_2^2}{4D_4}, \quad (75)$$

$$E_1 := -D_1 - \frac{2D_4 K_1 \left(K_2 + \frac{D_2}{2D_4}\right)}{1 - 2D_4 K_3^2}, \quad E_2 := -D_3 - \frac{D_4 K_1^2}{1 - 2D_4 K_3^2}, \quad (76)$$

with  $K_1, K_2, K_3$  as defined in (50), (51) and  $D_1, \dots, D_4$  in (68).

**Proof** Follows directly from the above results.

## 6 Numerical results

Any simulation scheme has to be tested, as they say the proof of the pudding is in the eating. In this section our goal is to test our proposed simulation schemes and compare them to alternate schemes. In our comparisons we focus on the bias of European call prices, where by bias we mean  $\mathbb{E}[\widehat{\alpha}] - \alpha$ , where  $\alpha$  is the true price of the European call and  $\widehat{\alpha}$  is its Monte Carlo estimator. It is of high importance for practitioners to have as small of a bias as possible for reasonable sizes of the time step. Ideally one would like to be able to simulate the relevant quantities only at those points in time which are relevant to the option contract that is being priced. Unfortunately, that is not always possible as has certainly become clear from several papers on the simulation of the Heston model, e.g. see Lord et al. (2008) or van Haastrecht and Pelsser (2009).

Table 1 contains the parameter configurations for our tests cases. Section 6.1 deals with the simulation scheme for the special case of the SZ model that collapses to a Heston model, see Section 3. Volatilities for Case I are similar to those in the equity market at the time of writing. In this test case we not only compare our scheme to an Euler scheme, but also to the best-performing scheme considered in Andersen (2008), the QE-M scheme. Finally, we also compare to a recently proposed scheme for the Heston model in Zhu (2008). In Case II we consider a setting of the SZ model which does not collapse to the Heston model. Here, the volatilities correspond to levels seen at the end of 2008 and beginning of 2009. Finally, we also test the scheme which was proposed in Section 5 for the SZHW model: case III deals with normal, perhaps slightly excited, long-term market volatilities.

Example	Type	$\kappa$	$\tau$	$\nu(0)$	$\theta$	$\rho_{xv}$	$r$	$a$	$\sigma$	$\rho_{xr}$	$\rho_{rv}$
case I	Call-5Y	0.1	0.3	0.0	0.0	-0.6	0.00	-	-	-	-
case II	Call-10Y	0.4	0.4	0.2	0.2	-0.9	0.04	-	-	-	-
case III	Call-15Y	0.4	0.4	0.2	0.2	-0.7	0.04	0.03	0.01	0.2	0.15

**Table 1:** Test cases for the Schöbel-Zhu/Heston, Schöbel-Zhu, and Schöbel-Zhu-Hull-White simulation scheme, in all cases  $S(0) = 100$ .

All numerical examples are based on a million simulation paths, where we used the stock price as a control variate and the Mersenne Twister to generate pseudo-random uniform numbers.

## 6.1 Results for the Heston/Schöbel-Zhu model

Starting with Case I, in which we consider a special case of the SZ model which corresponds to the Heston model, we can not only compare our Exponential Affine scheme (denoted by EAE) to a simple Euler scheme, but we can of course also compare it to the best-performing scheme of Andersen (2008), the QE-M scheme. Whereas our EAE scheme samples from the exact distributions of  $v(t)$  and  $v^2(t)$ , Andersen's QE scheme uses:

- a Quadratic Gaussian distribution when  $\text{Var}[v^2(t + \Delta)] / \mathbb{E}[v^2(t + \Delta)] \leq 1.5$ ,
- a mixture of zero and an exponential function, otherwise.

Our scheme and the QE-M scheme therefore differ for low values of  $v(t)$ , to be precise when:

$$v^2(t) \in \left[0, \frac{(e^{\kappa\Delta} - 1)\tau^2}{4\kappa}\right]. \quad (77)$$

Finally, we also compare to Zhu (2008) scheme. In this scheme, the SDE for the square root of the stochastic variance is derived. As the square root is not differentiable in zero, Ito's lemma is incorrectly applied here, e.g. see Kahl and Jäckel (2006) or Lord et al. (2008). Luckily, Zhu's best-performing method, a moment-matching method, does not depend too much on this premise. The numerical examples in his paper suggest that this method is comparable to Andersen's QE scheme for low values of the volatility of variance parameter, but is outperformed for realistic levels of the volatility of variance parameter.

	Euler	Zhu	QE-M	EAE
$\Delta$	$K = 100$			
1	-1.914 ( $\pm 0.062$ )	0.479 ( $\pm 0.075$ )	-0.059 ( $\pm 0.057$ )	-0.130 ( $\pm 0.057$ )
1/2	-0.781 ( $\pm 0.060$ )	0.291 ( $\pm 0.064$ )	-0.023* ( $\pm 0.057$ )	-0.020* ( $\pm 0.057$ )
1/4	-0.348 ( $\pm 0.058$ )	0.106 ( $\pm 0.060$ )	0.013* ( $\pm 0.057$ )	-0.039* ( $\pm 0.057$ )
1/8	-0.136 ( $\pm 0.058$ )	0.049* ( $\pm 0.059$ )	0.024* ( $\pm 0.057$ )	-0.003* ( $\pm 0.057$ )
1/16	0.002* ( $\pm 0.058$ )	0.041* ( $\pm 0.058$ )	-0.008* ( $\pm 0.057$ )	0.013* ( $\pm 0.057$ )
1/32	-0.015* ( $\pm 0.057$ )	0.037* ( $\pm 0.057$ )	-0.015* ( $\pm 0.057$ )	-0.010* ( $\pm 0.057$ )
	$K = 140$			
1	0.058* ( $\pm 0.074$ )	3.537 ( $\pm 0.099$ )	-0.524 ( $\pm 0.066$ )	-0.354 ( $\pm 0.066$ )
1/2	0.226 ( $\pm 0.071$ )	2.068 ( $\pm 0.078$ )	-0.147 ( $\pm 0.066$ )	-0.080 ( $\pm 0.066$ )
1/4	0.160 ( $\pm 0.068$ )	1.049 ( $\pm 0.071$ )	-0.006* ( $\pm 0.067$ )	-0.051* ( $\pm 0.066$ )
1/8	0.137 ( $\pm 0.067$ )	0.503 ( $\pm 0.068$ )	0.020* ( $\pm 0.066$ )	-0.012* ( $\pm 0.066$ )
1/16	0.138 ( $\pm 0.067$ )	0.251 ( $\pm 0.067$ )	-0.016* ( $\pm 0.066$ )	0.003* ( $\pm 0.066$ )
1/32	0.053* ( $\pm 0.066$ )	0.138 ( $\pm 0.066$ )	-0.038* ( $\pm 0.066$ )	-0.015* ( $\pm 0.066$ )
	$K = 60$			
1	-1.707 ( $\pm 0.036$ )	-0.991 ( $\pm 0.041$ )	0.066 ( $\pm 0.036$ )	-0.005* ( $\pm 0.036$ )
1/2	-0.793 ( $\pm 0.037$ )	-0.541 ( $\pm 0.038$ )	0.022* ( $\pm 0.036$ )	0.004* ( $\pm 0.036$ )
1/4	-0.381 ( $\pm 0.037$ )	-0.254 ( $\pm 0.037$ )	0.004* ( $\pm 0.037$ )	-0.017* ( $\pm 0.036$ )
1/8	-0.170 ( $\pm 0.037$ )	-0.114 ( $\pm 0.037$ )	0.013* ( $\pm 0.037$ )	0.003* ( $\pm 0.037$ )
1/16	-0.044 ( $\pm 0.037$ )	-0.034* ( $\pm 0.037$ )	0.005* ( $\pm 0.037$ )	0.004* ( $\pm 0.036$ )
1/32	-0.024* ( $\pm 0.037$ )	-0.004* ( $\pm 0.037$ )	-0.003* ( $\pm 0.037$ )	-0.001* ( $\pm 0.036$ )

**Table 2:** Estimated call option prices biases for case I. Numbers in parentheses are the widths of the confidence interval at a 99% confidence level: we starred the biases that were not significantly different from zero. Exact prices respectively are 27.90, 14.23 and 50.34.

In Table 2, we have displayed estimated call option price biases for Case I, as a function of the strike level ( $K = 60, 100$  or  $140$ ) and the time step  $\Delta$  (1 through  $1/32$ ). Numbers in parentheses are the widths of the confidence interval at a 99% confidence level: we starred the biases that were not significantly different from zero. From the results it is clear that - at least for this parameter configuration - Zhu's method is better than a simple Euler discretisation for lower strikes, though for higher strikes the Euler scheme wins. The QE-M and EAE methods however are much better in terms of bias. Both methods are too close to tell apart.

As one eventually wants to judge a scheme based on its efficiency, one should look its accuracy in combination with the computational effort of the methods. To this end we also report the computational times for the four simulation methods which are provided in Table 3 below.

$\Delta$	Euler	Zhu	QE-M	EAE
1	0.9	1.0	1.2	1.1
1/2	1.8	2.0	2.4	2.2
1/4	3.6	4.0	5.0	4.3
1/8	7.0	7.9	10.2	8.6
1/16	13.9	15.7	20.8	17.1
1/32	28.1	31.3	41.9	33.5

**Table 3:** Computational times in seconds for case I for the Euler, Zhu, QE-M and EAE scheme, all with one million sample paths.

From Table 3 we can see the Euler scheme takes the least time to compute, followed by Zhu's method, the EAE scheme and the QE-M scheme. Still, the efficiency of the QE-M scheme and the EAE method by far outperforms those of the Euler and Zhu's method as can be seen if we take a look at the accuracy of the methods in Table 2. From that table, we can see that the EAE and QE-M only need 2 or 4 time steps a year to produce of scheme with no significant bias, whereas the Euler and Zhu's scheme in most cases need at least 16 time steps a year to produce a scheme negligible bias. Though the QE-M scheme and the EAE method produce a similar accuracy, the EAE method is more efficient. This can be explained to the fact that the exact Gaussian volatility distribution of the Schöbel and Zhu (1999) model is explicitly utilized in the EAE method, whereas the variance simulation of QE-M method is tailored for Heston (1993) model.

## 6.2 Results for the Schöbel-Zhu and Schöbel-Zhu-Hull-White model

We move on to Case II and III, which are slightly more benign due to a non-zero value of  $\psi^6$  in Case II and the inclusion of stochastic interest rates for Case III. The numerical results for these cases can be found in Table 4 and 5. Computational times for both cases behave very similar to those reported in Table 3 and are hence omitted<sup>7</sup>.

In Case II, a non-Heston SZ model, the differences between the Euler and EAE methods are indeed closer, though still noticeably in favour of the EAE method. From Table 5 we can see that especially for in-and out-of-the-money options, the EAE scheme significantly outperforms the Euler scheme.

<sup>6</sup>This makes the distribution less fat-tailed.

<sup>7</sup>These are available upon request.

Finally, we take a look at the performance of the simulation schemes for the SZHW model, where in addition to the SZ model we have stochastic interest rates which are correlated with both the underlying and the stochastic volatility process. While the addition of stochastic interest rates complicates the scheme slightly, the picture is similar to before as can be seen from Table 6. Again the EAE method produces a much smaller discretisation error than the Euler scheme, allowing the user to utilise bigger time steps instead of the smaller ones one would be confined to when using the Euler method. For example, for the strikes considered one could safely use a timestep of a quarter of a year for the EAE method, and have a bias which is not significantly different from zero. In the Euler method, this is only achieved with a timestep equal to 1/16.

	Euler	EAE
$\Delta$	$K = 100$	
1	-0.828 ( $\pm 0.059$ )	-0.389 ( $\pm 0.050$ )
1/2	-0.314 ( $\pm 0.055$ )	-0.165 ( $\pm 0.050$ )
1/4	-0.145 ( $\pm 0.052$ )	-0.034* ( $\pm 0.050$ )
1/8	-0.068 ( $\pm 0.051$ )	0.014* ( $\pm 0.050$ )
1/16	-0.043* ( $\pm 0.051$ )	0.005* ( $\pm 0.050$ )
1/32	0.017* ( $\pm 0.050$ )	0.003* ( $\pm 0.050$ )
	$K = 140$	
1	-0.495 ( $\pm 0.080$ )	-0.457 ( $\pm 0.063$ )
1/2	-0.110 ( $\pm 0.072$ )	-0.204 ( $\pm 0.064$ )
1/4	-0.032* ( $\pm 0.068$ )	-0.041* ( $\pm 0.064$ )
1/8	-0.003* ( $\pm 0.066$ )	0.013* ( $\pm 0.064$ )
1/16	0.002* ( $\pm 0.065$ )	0.005* ( $\pm 0.064$ )
1/32	0.034* ( $\pm 0.065$ )	0.005* ( $\pm 0.064$ )
	$K = 60$	
1	-0.699 ( $\pm 0.035$ )	-0.278 ( $\pm 0.031$ )
1/2	-0.298 ( $\pm 0.033$ )	-0.110 ( $\pm 0.032$ )
1/4	-0.151 ( $\pm 0.033$ )	-0.023* ( $\pm 0.032$ )
1/8	-0.070 ( $\pm 0.032$ )	0.007* ( $\pm 0.032$ )
1/16	-0.046 ( $\pm 0.032$ )	0.004* ( $\pm 0.032$ )
1/32	0.005* ( $\pm 0.032$ )	0.004* ( $\pm 0.032$ )

**Table 4:** Results for case II, exact prices: 56.77, 45.34 and 70.89.

	Euler	EAE
$\Delta$	$K = 100$	
1	-0.323 ( $\pm 0.032$ )	-0.069 ( $\pm 0.032$ )
1/2	-0.151 ( $\pm 0.032$ )	-0.036 ( $\pm 0.032$ )
1/4	-0.087 ( $\pm 0.032$ )	0.008* ( $\pm 0.032$ )
1/8	-0.029* ( $\pm 0.032$ )	0.005* ( $\pm 0.032$ )
1/16	-0.006* ( $\pm 0.032$ )	-0.022* ( $\pm 0.032$ )
1/32	0.001* ( $\pm 0.032$ )	0.001* ( $\pm 0.032$ )
	$K = 140$	
1	-0.299 ( $\pm 0.047$ )	-0.085 ( $\pm 0.046$ )
1/2	-0.127 ( $\pm 0.046$ )	-0.046* ( $\pm 0.046$ )
1/4	-0.077 ( $\pm 0.046$ )	0.012* ( $\pm 0.046$ )
1/8	-0.014* ( $\pm 0.046$ )	0.006* ( $\pm 0.046$ )
1/16	-0.002* ( $\pm 0.046$ )	-0.022* ( $\pm 0.046$ )
1/32	0.014* ( $\pm 0.046$ )	-0.008* ( $\pm 0.046$ )
	$K = 60$	
1	-0.190 ( $\pm 0.016$ )	-0.041 ( $\pm 0.017$ )
1/2	-0.095 ( $\pm 0.017$ )	-0.017 ( $\pm 0.017$ )
1/4	-0.052 ( $\pm 0.017$ )	0.005* ( $\pm 0.017$ )
1/8	-0.019 ( $\pm 0.017$ )	0.005* ( $\pm 0.017$ )
1/16	0.005* ( $\pm 0.017$ )	-0.007* ( $\pm 0.017$ )
1/32	-0.001* ( $\pm 0.017$ )	0.004* ( $\pm 0.017$ )

**Table 5:** Results for case III, exact prices: 53.75, 40.69 and 69.97.

Estimated call option prices biases for case II and III. Numbers in parentheses are the widths of the confidence interval at a 99% confidence level: we starred the biases that were not significantly different from zero.

Finally, we take a look at the performance of the simulation schemes for the SZHW model, where in addition to the SZ model we have stochastic interest rates which are correlated with both the underlying and the stochastic volatility process. While the addition of stochastic interest rates complicates the scheme slightly, the picture is similar to before as can be seen from Table 5. Again the EAE method produces a much smaller discretisation error than the Euler scheme, allowing the user to utilize bigger time steps instead of the smaller ones one would be confined to when using the Euler method.

## 7 Conclusion

A major problem signaled with Euler schemes in the simulation of stochastic volatility models is their inability to generate the proper correlation between the increments of the asset and the stochastic volatility processes. As the correlation parameter in the stochastic volatility models is an important determinant of the skew in implied volatilities, not being able to match this parameter leads to a significant mispricing of options with strikes far away from the at-the-money level. In the [Heston \(1993\)](#) model, this so-called "leaking correlation" problem, is partially caused by the fact that an Euler discretisation tries to approximate a square root process, by a Gaussian process. However even when the stochastic volatility itself is Gaussian, such as in [Schöbel and Zhu \(1999\)](#)-like models, we have shown that the problem of "leaking correlation" is still an issue.

In this paper we have proposed simulation algorithms for the SZ model and its extensions. By analyzing the lessons learned on how to avoid the so-called leaking correlation phenomenon in the simulation of the [Heston \(1993\)](#) model, we formulated a simulation scheme for the SZ model which is tailored to match the correlation between the increments of the asset price and the variance processes of the continuous-time dynamics. A simulation scheme for the Schöbel-Zhu-Hull-White model considered in [van Haastrecht et al. \(2009\)](#), which incorporates the need for stochastic interest rates, was derived as well. This is closely related to the recent advances in the development of markets for long-term derivatives, for which maturities the inclusion of stochastic interest rates in a derivatives pricing model is more appropriate.

All introduced schemes have carefully been chosen to be Exponentially Affine in Expectation (EAE), which greatly facilitates the derivation of a martingale correction. The regularity of both schemes has also been studied. Finally, we numerically compared the new simulation schemes to other recent schemes in the literature. For a special case of the SZ model which coincides with the Heston model, our proposed scheme has a similar performance to the QE-M scheme of [Andersen \(2008\)](#), whilst being slightly more efficient in terms of computational time required. For all non-Heston SZ and the SZHW model, it has been demonstrated that our scheme consistently outperforms the Euler scheme. These results affirm that Andersen's result is more widely applicable than to the Heston model alone; for the simulation of stochastic volatility models, it is of great importance to match the correlation between the asset price and its stochastic volatility process.

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